Imperial College London

M4R Research Project

Imperial College London

Department of Mathematics

Galois and Tannakian Categories

Author:
Ned Summers
01073391

Supervisor:
Prof. Johannes Nicaise

April 3, 2020

This is my own unaided work unless stated otherwise:

Abstract

As with all good mathematics, category theory attempts to create connections between abstract ideas. A way of translating information between different mathematical ideas provides free theorems and proofs but far greater than this, it gives us a means to formalise similarities in intuition or indeed form entirely new intuition, useful whether or not an idea is well understood. Category theory does this in such a general way that it may almost been seen as a language. It is only in this language that connections that appear to us intuitively can be made into formal mathematical correspondences. In this project, we will deal with the preliminaries of category theory, with a particular focus on forming examples in the categories of covering spaces for a topological space X, and separable field extensions. It is here we will see how the language of category theory connects these seemingly very separate areas of maths as instances of Galois Categories. With our categorical foundations laid, we will cover an introduction to the theory of affine group schemes and lay out the definition of a neutral Tannakian categories to see how these two areas are related.

Contents

Introduction **4**

1 Categories, Diagrams and a New Approach **5**
- 1.1 Categories . . . 5
- 1.2 Digression on Diagrams . . . 9
- 1.3 Creating New Categories from Old . . . 10
- 1.4 From Objects to Morphisms . . . 11
- 1.5 The Category of Separable Algebras over a Field . . . 15

2 Functors, Natural Transformations and Limits **17**
- 2.1 Functors . . . 17
- 2.2 A New Perspective on Diagrams . . . 20
- 2.3 Natural Transformations . . . 20
- 2.4 Equivalences . . . 22

3 Limits and Colimits **25**
- 3.1 Cocones and Colimits . . . 25
- 3.2 Cones and Limits . . . 28
- 3.3 Fibre Products in Familiar Categories . . . 30
- 3.4 Inverse Limits and Profinite Groups . . . 32

4 Galois Categories **33**
- 4.1 The Definition of a Galois Category . . . 33
- 4.2 Example: Finite Sets . . . 36
- 4.3 Example: Covering Spaces . . . 36
- 4.4 Example: Separable Algebras . . . 38
- 4.5 Properties . . . 39
- 4.6 Concluding Galois Categories . . . 43

5 Affine Group Schemes **45**
- 5.1 The Definition of Affine Group Schemes . . . 45
- 5.2 Reconsidering Groups . . . 47
- 5.3 Yoneda's Lemma and Hopf Algebras . . . 47
- 5.4 Representations of an Affine Group Scheme . . . 50

6 Tannakian Categories **51**
- 6.1 Tensor Categories . . . 51
- 6.2 Abelian Tensor Categories . . . 52
- 6.3 Rigid Tensor Categories . . . 53
- 6.4 Tensor Functors . . . 54
- 6.5 Neutral Tannakian Categories and a Final Equivalence . . . 55

Conclusion **56**

A Appendix **57**
A.1 Separable algebras . . . 57
A.2 Galois Field Theory . . . 58
A.3 Group Actions . . . 59
A.4 Yoneda for Separable Algebras . . . 59

Introduction

In the study of algebraic topology, when students first meet fundamental groups they are commonly introduced to the topic of covering spaces. The theory is geometrically intuitive but also surprising and revealing. The relationships between deck transformations and fundamental groups are a good example of how topological information corresponds to algebraic information. In an undergraduate course, the conclusion of that section is often the correspondence between isomorphism classes of certain covering spaces and sets of subgroups of the fundamental group.

At a similar level of study, students might take a course on Galois theory. It is most likely in a different part of the maths department, but towards the end similarities seem to arise. Again there is a large object existing "over" a small one, of automorphisms which preserve structure. In the main theorem, we are introduced to another correspondence, this time between field extensions and open subgroups of a Galois group.

There is an underlying structure that links these two examples, and it is best expressed in the language of category theory. The similarities are clearest in the form of the following two theorems.

Theorem 0.0.1. *Let k be a field. Then the category ${}_k\mathbf{SAlg}$ of separable k-algebras is anti-equivalent to the category $\mathrm{Gal}(k_s/k)$-$\mathbf{FinSet}$ of finite sets with a continuous action of $\mathrm{Gal}(k_s/k)$, where k_s is the separable closure of k.*

Theorem 0.0.2. *Let X be a topological space. Then there is a canonical profinite group $\hat{\pi}(X, x)$ for any $x \in X$ such that the category $\mathbf{Cov}_X$ of finite covering spaces of X is equivalent to the category $\hat{\pi}(X, x)$-$\mathbf{FinSet}$ of finite sets with continuous action of $\hat{\pi}(X, x)$.*

Further, if X admits a universal cover, then $\hat{\pi}(X, x)$ is exactly the profinite completion of $\pi(X, x)$, the fundamental group of X. If the cover is finite, then this is exactly $\pi(X, x)$.

The first half of this paper is a comprehensive exposition on the fundamentals of category theory. We rely heavily on examples and in particular emphasise the shift in focus from the contents of objects to the morphisms between them that this areas of maths espouses. The aim is to introduce the reader to the way that category theory generalises and pulls together many fields, with the hope that the intuition developed will help us when we work with more complicated categories and proofs.

In this part, we focus closely on two examples of categories, $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}^{\mathrm{op}}$. These are less clean cut than more familiar ones and allow us to look at how the ideas we generalise with categorical language apply in such settings.

This focus also has a dual purpose. In chapter 4, we utilise our newly developed tools to approach the statements of theorem 0.0.1 and theorem 0.0.2, which make strong claims about the two familiar categories. The aim of this chapter is to introduce the definition of a Galois category and to show that exactly these give rise to the algebraic behaviour in the two theorems.

The final section of the project looks at another correspondence between a category defined entirely by its properties and an algebraic ones. We give a brief introduction to the theory of affine group schemes, including their representations, and then give a similarly brief introduction to tensor categories. Our final result is the statement of a surprising equivalence between the categories of representations of an affine group scheme and a very particular tensor category, called a neutral Tannakian category.

The results from the first part draw largely from [2] and [6]. On Galois categories, [6] remains an important source alongside [5] and [12]. The last section follows closely [3] and [11]. Since the main focus of this work is pedagogical, many of the examples are standard but the perspectives explored are our own. Similarly, there are a number of sources that occur only once or twice in the pursuit of a good example or revealing definition.

There is only one piece of mathematical convention that needs stating: all rings will assumed to be commutative with 1.

Chapter 1

Categories, Diagrams and a New Approach

1.1 Categories

Unless stated otherwise, the definitions in this section are those in [2].

Definition 1.1.1. A *category* $\mathbf{C}$ consists of the following data:

1. a collection, $\mathrm{Ob}\,(\mathbf{C})$, of *objects*: A, B, C, ...
2. for each pair of objects A and B, a collection $\mathbf{C}(A, B)$, or $\mathrm{Hom}_{\mathbf{C}}(A, B)$, of *morphisms*: f, g, h, ...

with two further pieces of information:

(i) for each triple of objects A, B and C, *a composition law*

$$\begin{aligned} - \circ -\colon \mathrm{Hom}_{\mathbf{C}}(A, B) \times \mathrm{Hom}_{\mathbf{C}}(B, C) &\to \mathrm{Hom}_{\mathbf{C}}(A, C) \\ (f, g) &\mapsto g \circ f = gf \end{aligned}$$

which is associative: if $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ and $h\colon C \longrightarrow D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{1.1}$$

(ii) for each object A, there is a designated *identity morphism* $\mathbf{1}_A\colon A \longrightarrow A$ with the property that for any object B and any $f\colon A \longrightarrow B$,

$$\mathbf{1}_B \circ f = f = f \circ \mathbf{1}_A$$

Remark. Since some of the morphisms we look at will not be functions, we will denote a morphism with a longer arrow than a function, like $f\colon A \longrightarrow B$. Some of the language of functions, however, is adopted here:

- Morphisms may also referred to as *maps* or *arrows*.
- If $f\colon A \longrightarrow B$, then $\mathrm{dom}(f) = A$ is the *domain* of f and $\mathrm{cod}(f) = B$ is the *codomain* of f.
- Where there is no ambiguity, we may write gf for $g \circ f$.

Definition 1.1.2. A category $\mathbf{C}$ is *small* if and only if $\mathrm{Ob}\,(\mathbf{C})$ is a set and for all $A, B \in \mathrm{Ob}\,(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is also a set.

This definition hints that we may talk about categories that do not fulfil these conditions, but for the moment let us look at some examples of small categories.

Examples. 1. Since morphisms were a generalisation of set-functions, a good first example of a category is any universe of sets and the functions between them.

For example, we may construct a category $\mathbf{N}$ which has $\mathrm{Ob}(\mathbf{N}) = \mathcal{P}(\mathbb{N})$, the power-set of the set of natural numbers, so the objects of $\mathbf{N}$ are just subset of $\mathbb{N}$. Given two objects, $N, M \subset \mathbb{N}$, we can take $\mathrm{Hom}_{\mathbf{N}}(N, M)$ to be the set of all functions $f\colon N \to M$. Composition is given by function composition and the identity morphism is the identity function.

Alternatively, let $\mathbf{N}'$ have the same set of objects, but now $\mathrm{Hom}_{\mathbf{N}'}(N, M)$ has only increasing functions $f\colon N \to M$. Since the composition of increasing functions is increasing, and identity map on any set is itself increasing, this satisfies all the conditions required.

2. Many of the most commonly seen categories have as objects some kind of set with an algebraic structure and have structure preserving maps as morphisms.

We then might invent the category **fsgrp**, of finite symmetric groups, has as objects the groups S_r of permutations of the set $\{1, \ldots, r\}$ for $r \in \mathbb{N}$. The morphisms $\mathrm{Hom}_{\mathbf{fsgrp}}(S_r, S_n)$ are the homomorphisms $\phi\colon S_r \to S_n$.

3. [7] Objects need not be sets and morphisms need not be functions. Indeed the generality of these axioms allow a whole lot of flexibility as to what can be seen as a category.

Recall. A *partially ordered set* is a pair $(X, \leqslant)$ with a set X and $\leqslant$ a reflexive, anti-symmetric, transitive relation on X.

From any partially order set $(X, \leqslant)$ we can construct a category $\mathbf{X}$ that contains exactly the same information. This category has $\mathrm{Ob}(\mathbf{X}) = X$ and we have that $\mathrm{Hom}_{\mathbf{X}}(x, y)$ has exactly one element if and only if $x \leqslant y$ and is empty otherwise.

Note that we can only consider the composition of morphisms $f_{xy}\colon x \longrightarrow y$ and $f_{yz}\colon y \longrightarrow z$ if there exist such morphisms. By our definition, this is equivalent to saying that $x \leqslant y$ and $y \leqslant z$. If we have this then the composition is $f_{yz} \circ f_{xy}\colon x \longrightarrow z$. Since $\leqslant$ is transitive, we must have that $x \leqslant z$ and so there is exactly the one such morphism. Hence we define composition exactly as such.

Since $x \leqslant x$, there is exactly one morphism $f_{xx}\colon x \longrightarrow x$ and exhibits exactly what is needed to be the identity morphisms for if there is a morphism $f_{xy}\colon x \longrightarrow y$ then it is the unique element of $\mathrm{Hom}_{\mathbf{X}}(x, y)$ and $f_{xy} \circ f_{xx}\colon x \longrightarrow y$ must again be this unique morphism.

Note that some authors take the convention that $\mathrm{Hom}_{\mathbf{X}}(x, y)$ has only one element if and only if $y \leqslant x$, but the method is otherwise the same.

4. [2] Similarly, we view a group as a specific type of category. Let G be a group and let us define a category $\mathbf{G}$ by giving it a single object, which we will call $*$ and letting $\mathrm{Hom}_{\mathbf{G}}(*, *)$ (the only collection of morphisms in this category) consist of exactly the elements of the group G. Composition of morphisms is given by multiplication of the elements in G.

The associativity condition of group multiplication gives associativity of composition and the identity element of the group corresponds to the identity morphism $\mathbf{1}_*$.

There are a number of ways to develop the foundations of category theory such that we are not constrained to small categories. This is helpful, indeed large categories are often more intuitive and appear more often. For example, **Set**, *the category of sets*, consisting of all sets as objects and all possible functions between two sets as the arrows between them is not a small category, since $\mathrm{Ob}(\mathbf{Set})$ is not a set, but we will find plays a crucial part in our later work. These foundations are a large topic that deserves their own project and so we shall largely ignore them, taking for granted instead that we can talk meaningfully about categories that are in this sense *large*. A good discussion of how this may be done can be found in sections 1.6-7 of [7].We will adopt some set-theoretical language for our shorthand: We will use $A \in \mathrm{Ob}(\mathbf{C})$ to mean A is an object of the category $\mathbf{C}$ and will write $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ for $f\colon A \longrightarrow B$, even if these are not strictly sets.

Despite all this, we might want to talk about categories that, even though they are not small, do have some constraint on size:

Definition 1.1.3. A category $\mathbf{C}$ is called *locally small* if and only if, for all $A, B \in \mathrm{Ob}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is a set.

All small categories are locally small. The category **Set** defined above is also locally small. Even though there is no set of all sets, given sets A and B there is always a set of all functions from A to B.

Now we have broken from the constraints of small categories we can find many more categories which we have encountered before:

Examples.

1. If we don't like infinite sets, we instead might consider **FinSet**, the category of finite sets, which has finite sets as its objects and the functions between them as morphisms. In fact, this category will be used heavily in chapter 4.

2. As we saw before, a huge number of commonly encountered categories fall under categories of objects that are sets with some structure with morphisms being functions that relate to this structure in some way:

Category name	Category objects	Category morphisms
Grp	Groups	Group homomorphisms
AbGrp	Abelian Groups	Group homomorphisms
Rng	Rings	Ring homomorphisms
Top	Topological Spaces	Continuous maps
$\mathbf{Top}_*$	Pointed Topological Spaces	Point-preserving continuous maps
Htpy	Topological Spaces	Homotopy classes of continuous maps

It is worth noting that **Top** and **Htpy** have the same objects but the morphisms are different. The definition of a category is flexible enough to allow this. In fact, as we go on here we will realise that the majority of the 'good stuff' only really refers to objects in passing, with the actual category theory concerning itself mostly with morphisms. It is this shift in perspective that can be so powerful: if we are considering sets, we are most often concerned with their elements, whereas if we are considering categories, we are more interesting in how the objects relate to one another through arrows.

3. An example that will be crucial later on in this paper requires some background from Algebraic topology:

Definition 1.1.4. Let X be a topological space. Given a map $f\colon Y \to X$, an open set $U \subset X$ is *evenly covered by* f if $f^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeomorphically onto U by f. In other words: $f^{-1}(U) = \bigcup_{i \in I} V_i$ with $V_i \cap V_j = \varnothing$ if i $i \neq j$ and further, for all $i \in I$, $f|_{V_i} \colon V_i \to U$ is a homeomorphism.

A covering space of X is a continuous map $p\colon Y \to X$ for some space Y with the property that for all $x \in X$ there is a neighbourhood of x that is evenly covered by p. [4]

A lot of the studying spaces and their covers, as we would expect, needs to consider all the points 'above' a point in the base. So, for a covering space $p\colon Y \to X$ of X and a point $x \in X$, *the fibre over* x is the set $p^{-1}(\{x\})$, though we may abuse notation and write $p^{-1}(x)$. A *finite covering space of* X is a covering space where all the fibres are finite.

A helix, with the map that forgets the height of a point and instead projects it directly onto $\mathbb{S}^1$ is a covering space. Unfortunately the scope of this paper doesn't allow us to indulge in examples here, but intuitively we can imagine a covering space as the result of cutting and gluing a number of copies of X in a symmetric enough way. The helix is the result of taking $\bigsqcup_{n \in \mathbb{N}} \mathbb{S}^1$, lining each copy of $\mathbb{S}^1$ in a stack, cutting them all at a point and sticking one of the resulting lose ends to the copy above and one below.

If we want to study finite covering spaces with category theory, so we must construct an appropriate category. Let us fix a topological space X. We might be inclined to take the objects of our new category $\mathbf{Cov}_X$ to be those topological spaces Y that omit a finite covering space $p\colon Y \to X$. With a little investigation, however, we notice that Y might be part of a covering space in more than one way: $\mathbf{1}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is a covering space (all identity arrows are) but so are all reflections and rotations of $\mathbb{S}^1$. We can see a covering map $p\colon Y \to X$ as encoding a way in which Y can be folded onto X. For a given Y we would hope to distinguish between the different foldings.

The solution is to take the objects of $\mathbf{Cov}_X$ to in fact be the foldings themselves, the finite covering spaces $p\colon Y \to X$ for any topological spaces Y. Our next step then is to decide on morphisms. Given two covering spaces $p_Y : Y \to X$ and $p_Z\colon Z \to X$, we define morphisms from

p_Y to p_Z to be continuous maps $\phi : Y \to Z$ with the additional property of preserving the folding onto X: from a point $y \in Y$, it doesn't matter if we go straight to X or we go over ϕ to Z first. We may express this directly by

$$\mathrm{Hom}_{\mathbf{Cov}_X}(p_Y : Y \to X, p_Z : Z \to X) = \{\phi : Y \to Z \mid p_Z \circ \phi = p_Y\}.$$

This relationship can also be shown by an illustration:

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "p_Y"'] & & Z \arrow[dl, "p_Z"] \\ & X & \end{tikzcd}$$

The morphisms are exactly those continuous maps ϕ such that if we take an element of Y, it wouldn't matter the route of arrows we take around this diagram.

We compose morphisms by regular function composition. If $p_W : W \to X$, $p_Y : Y \to X$ and $p_Z : Z \to X$ are all covering spaces, $\phi_1 : W \to Y$ and $\phi_2 : Y \to Z$ are covering space morphisms, then

$$\phi_2 \circ_{\mathbf{Cov}_X} \phi_1 = \phi_2 \circ \phi_1$$

since

$$p_Z \circ (\phi_2 \circ \phi_1) = (p_Z \circ \phi_2) \circ \phi_1 = p_Y \circ \phi_1 = p_W$$

as we'd hope for a morphism from p_W to p_Z.

This can be seen instead using our illustrations:

We have

$$\begin{tikzcd} W \arrow[r, "\phi_1"] \arrow[dr, "p_W"'] & Y \arrow[d, "p_Y"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\phi_2"] \arrow[d, "p_Y"'] & Z \arrow[dl, "p_Z"] \\ X & \end{tikzcd}$$

and since these diagrams tells us routes that are permissible, we can join them at the common map in the middle p_Y to get a new illustration:

$$\begin{tikzcd} W \arrow[r, "\phi_1"] \arrow[rr, bend left, "\phi_2 \circ \phi_1"] \arrow[dr, "p_W"'] & Y \arrow[r, "\phi_2"] \arrow[d, "p_Y"] & Z \arrow[dl, "p_Z"] \\ & X & \end{tikzcd}$$

where we can see the composition $\phi_2 \circ \phi_1$ is allowed because it is just the action of following the top two arrows. Since this is telling us permissible routes, we are also ok to forget arrows. So forgetting all the arrows in the middle, we have

$$\begin{tikzcd} W \arrow[rr, "\phi_2 \circ \phi_1"] \arrow[dr, "p_W"'] & & Z \arrow[dl, "p_Z"] \\ & X & \end{tikzcd}$$

which is exactly the diagram we needed to show $\phi_2 \circ \phi_1$ is a morphisms of covering spaces.

Associativity and identity are adopted directly because these are continuous maps. Thus we have constructed our last example of a category, the category $\mathbf{Cov}_X$ over covering spaces over a topological space X.

Formally, then:

Definition 1.1.5. Let X be a topological space. *The category of finite coverings of* X, denoted $\mathbf{Cov}_X$, has as its objects covering spaces $p_Y : Y \longrightarrow X$ and for morphisms $\mathbf{Cov}_X(p_{Y_1}, p_{Y_2})$, continuous maps $\phi : Y_1 \longrightarrow Y_2$ such that $p_{Y_2} \circ \phi = p_{Y_1}$.

1.2 Digression on Diagrams

In our discussion of diagrams and illustrations we have happened upon an idea essential to categorical thinking and which will feature heavily in our future arguments: commutative diagrams.

Definition 1.2.1. A *diagram in a category* $\mathbf{C}$ is a directed multigraph (a graph where multiple edges are allowed between vertexes) where the vertexes are objects of $\mathbf{C}$ and the edges are morphisms between the corresponding edges, in the correct direction. Further, we say *the diagram commutes* or it is *a commutative diagram* if any two paths between edges are the same, where the process of following one edge and then another is given by composing these morphisms.

We will see this is an unnecessarily restricted definition of a diagram and supersede it in section 2.2. Nonetheless, this definition is sufficient for the moment and gives us intuition that will help further along.

Example. Let A, B and C be sets, and let us take functions $f\colon A \to B$, $g\colon B \to C$ and $h\colon A \to C$. Then

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is a diagram in **Set** and it commutes if and only if $h = g \circ f$.

A lot of the information we have already discussed can be restated in terms of diagrams:

Examples. 1. The associativity of composition of morphisms given in eq. (1.1) can be expressed as the statement: for any category $\mathbf{C}$, objects $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$ and morphisms $f\colon A \longrightarrow B, g\colon B \longrightarrow C, h\colon C \longrightarrow D$ the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & B \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D. \end{tikzcd}$$

We will use this property without concern in all our diagrams.

2. The key property of identity morphisms can be expressed as the statement: for any category $\mathbf{C}$, objects $A, B \in \mathrm{Ob}(\mathbf{C})$ and morphism $f\colon A \longrightarrow B$ the following diagram commutes

$$\begin{tikzcd} A \arrow[loop left, "\mathbf{1}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathbf{1}_B"] \end{tikzcd}.$$

3. Let us give a more particular example in **Set**. Let A, B and C be sets, and let $A \times B$ be the Cartesian product with projections π_A and π_B. Then for any functions $f_A\colon C \to A$ and $f_B\colon C \to B$, the following diagram commutes

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B. \\ & C \arrow[ul, "f_A"] \arrow[u, dashed, "(f_A, f_B)"] \arrow[ur, "f_B"'] & \end{tikzcd}$$

Our next important definition then can be constructed with the help of diagrams.

Definition 1.2.2. A morphism $f\colon A \longrightarrow B$ in a category $\mathbf{C}$ is *an isomorphism* if there exists an morphism $f^{-1}\colon B \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[loop left, "\mathbf{1}_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f^{-1}"] \arrow[loop right, "\mathbf{1}_B"] \end{tikzcd}$$

That is, such that f^{-1} is an inverse to f:

$$f \circ f^{-1} = \mathbf{1}_B \text{ and } f^{-1} \circ f = \mathbf{1}_A.$$

We may denote an isomorphism by writing $f\colon A \xrightarrow{\sim} B$, and will say that A and B are *isomorphic*, denoted by $A \cong B$.

Examples. 1.

Category	Category morphisms	Category isomorphisms
Set	Set functions	Bijections
Grp	Group homomorphisms	Group isomorphisms
Rng	Rings homomorphisms	Ring isomorphisms
Top	Continuous functions	Homeomorphisms
Htpy	Homotopy classes of continuous maps	Homotopy Equivalences [8]

2. If we return to our example of a group G as a category $\mathbf{G}$ with a single element $*$ and with $\mathrm{Hom}_{\mathbf{G}}(*, *) = G$ then we note that the fact that all elements of the group have an inverse is equivalent to saying that every morphism in $\mathbf{G}$ is an isomorphism.

   A category where every morphisms is an isomorphism is called *a groupoid.*

1.3 Creating New Categories from Old

Now that we have some definitions, and some examples of categories, we can look at examples of how to build new categories from old ones.

Definition 1.3.1. Given a category $\mathbf{C}$ we can construct *the opposite category*, denoted $\mathbf{C}^{\mathrm{op}}$, which is the result of just reversing all the arrows in $\mathbf{C}$.

In other words, $\mathrm{Ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A)$ and composition is given by

$$\phi \circ_{\mathbf{C}^{\mathrm{op}}} \psi :\, = \psi \circ_{\mathbf{C}} \phi.$$

This gives rise to an idea of duality. If a proposition is true for the category $\mathbf{C}$ then we can derive a proposition that is true for $\mathbf{C}^{\mathrm{op}}$ by reversing all the arrows in the original proposition. In the simplest case, if we know a diagram commutes in $\mathbf{C}$ then the diagram in $\mathbf{C}^{\mathrm{op}}$ obtained by reversing all the arrows must also commute. Often, we use the prefix *co-* to denote an construction performed in the opposite category, or in some sense with the arrows reversed.

Example. For a group considered as a category with only one object $\mathbf{G}$, the opposite category $\mathbf{G}^{\mathrm{op}}$ is exactly the opposite group[1]. If $\mathbf{G}$ is commutative, this is the same as saying for any morphisms $\phi, \psi \in \mathrm{Hom}_{\mathbf{G}}(*, *)$ the diagram

$$\begin{tikzcd} * \arrow[r, "\phi"] \arrow[d, "\psi"'] & * \arrow[d, "\psi"] \\ * \arrow[r, "\phi"'] & * \end{tikzcd}$$

commutes. Correspondingly, for any two morphisms $\phi^{\mathrm{op}}, \psi^{\mathrm{op}} \in \mathrm{Hom}_{\mathbf{G}^{\mathrm{op}}}(*, *)$, the opposite diagram

$$\begin{tikzcd} * & * \arrow[l, "\phi^{\mathrm{op}}"'] \\ * \arrow[u, "\psi^{\mathrm{op}}"] & * \arrow[l, "\phi^{\mathrm{op}}"] \arrow[u, "\psi^{\mathrm{op}}"'] \end{tikzcd}$$

commutes. Which says exactly that the opposite group is also commutative. This is slight overkill though: if G is commutative, then $G = G^{\mathrm{op}}$.

Definition 1.3.2. Given a category $\mathbf{C}$, a category $\mathbf{D}$ is *a subcategory of* $\mathbf{C}$ if and only if every object of $\mathbf{D}$ is also an object of $\mathbf{C}$ and for any $A, B \in \mathrm{Ob}(\mathbf{D})$, $\mathrm{Hom}_{\mathbf{D}}(A, B) \subset \mathrm{Hom}_{\mathbf{C}}(A, B)$. (We again ignore that these may not be sets.)

If, for all $A, B \in \mathrm{Ob}(\mathbf{D})$ we have, in fact, that $\mathrm{Hom}_{\mathbf{D}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$ then $\mathbf{D}$ is called *a full subcategory of* $\mathbf{C}$.

Examples. 1. The category whose objects are $\mathcal{P}(\mathbb{N})$ and whose morphisms are increasing functions, is a subcategory of **Set**. It is not full, since in **Set** decreasing functions are also morphisms.

2. **AbGrp** is a full subcategory of **Grp**.

3. For a group $\mathbf{G}$ considered as a category, any subgroup of $\mathbf{G}$ is a subcategory. Not every subcategory of $\mathbf{G}$ defines a subgroup, however, because they need to contain inverse morphisms.

[1]The group with the same elements and the operation $g \cdot_{\mathrm{op}} h = h \cdot g$

Our last construction shows again how we are trying to move away from our study of objects themselves and instead moving our attention to the behaviour of morphisms. Particularly, to study an object in a category, it may be useful to study the morphisms for which it is the codomain.

Definition 1.3.3. Let $\mathbf{C}$ be a category and let $B \in \mathrm{Ob}(\mathbf{C})$. *The Slice category* $\mathbf{C}/B$ *of the* $\mathbf{C}$ *over* B has as its objects all the morphisms in $\mathbf{C}$ that have codomain B. Objects in this category thus have the form $f: X \longrightarrow B$ for some $X \in \mathrm{Ob}(\mathbf{C})$.

So what are is $\mathrm{Hom}_{\mathbf{C}/B}(f: X \longrightarrow B, g: Y \longrightarrow B)$? The obvious choice is some selection of morphisms in $\mathrm{Hom}_{\mathbf{C}}(X, Y)$. Since these two maps in some way put X and Y in B, we choose our morphisms to preserve this. In other words, arrows from f to g are morphisms $\phi: X \longrightarrow Y$ such that

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

commutes.

As we've seen in the covering spaces example, these triangles compose happily by function composition.

Example. It is not a coincidence that these triangles also appeared when we constructed $\mathbf{Cov_X}$ for a topological space X. $\mathbf{Cov_X}$ is a full subcategory of the slice category $\mathbf{Top}/X$. Note that not all the objects of $\mathbf{Top}/X$ are in $\mathbf{Cov_X}$, since not every continuous map $f: Y \to X$ defines a covering map.

This is also a perfect opportunity to see an example of a "co-" construction. Given the definition of the slice category, $\mathbf{C}/B$, we can define the coslice category $B/\mathbf{C}$ by looking at arrows going in the opposite direction.

Definition 1.3.4. Let $\mathbf{C}$ be a category and let $B \in \mathrm{Ob}(\mathbf{C})$. *The Coslice category* $B/\mathbf{C}$ *of* $\mathbf{C}$ *under* B has as its objects all the morphisms in $\mathbf{C}$ that have domain B. Objects in this category thus have the form $u: B \longrightarrow X$ for some $X \in \mathrm{Ob}(\mathbf{C})$.

The morphisms $\mathrm{Hom}_{B/\mathbf{C}}(u: B \longrightarrow X, v: B \longrightarrow Y)$ are then $\alpha: X \longrightarrow Y$ such that

$$\begin{tikzcd} & B \arrow[dl, "u"'] \arrow[dr, "v"] & \\ X \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

commutes.

Example. Let us quickly consider what the category $\{*\}/\mathbf{Top}$ looks like, with $\{*\}$ a discrete space of one element.

The elements of this category are maps $u: \{*\} \to X$ for topological spaces X. Each of these maps pick out a specific element of the space they map to. Let us write $x_u = u(\{*\}) \in X$ for the element of X that u chooses.

What are the morphisms in this category? Let us take $u: \{*\} \to X$ and $v: \{*\} \to Y$. Then for a function $\alpha: X \to Y$ to be a morphism between these two arrows in $\{*\}/\mathbf{Top}$ we must have

$$\alpha \circ u = v$$

which means exactly that

$$\alpha(x_u) = \alpha \circ u(*) = v(*) = x_v$$

and thus a continuous map α is in $\mathrm{Hom}_{\{*\}/\mathbf{Top}}(u, v)$ if and only if it preserves the designated point. In other words, $\{*\}/\mathbf{Top}$ is isomorphic (a term we will give a precise definition to in section 2.1) to $\mathbf{Top}_*$, the category of pointed topological spaces.

1.4 From Objects to Morphisms

We are now in a position where we can start to really focus on the first change in perspective that categorisation allows us. In this section, we are going to focus on generalising constructions, from injective functions to products of sets, in a way that will allow us to stop trying to study the elements of objects and instead turn our focus to morphisms. This allows us to study these constructions in categories where the objects are not sets, or where the morphisms don't look like set functions. From here, our definitions are drawn from [6] unless stated otherwise.

1.4.1 Injectivity and surjectivity

When we define injectivity and surjectivity for the first time, it is always done within the context of how they act on elements.

Recall. A function $f\colon X \to Y$ is *injective* if and only if for all $a, b \in X$,

$$f(a) = f(b) \implies a = b.$$

It is *surjective* if and only if for all $y \in Y$, there is $a \in A$ such that $f(a) = y$.

In an intuitive way, injections remember where the elements from the domain go, whilst surjections may mix up the domain, but they see everything in the range. The set of elementary propositions we prove about these functions included two equivalent definitions.

Recall. A function $f\colon X \to Y$ is injective if and only if for any functions $\phi, \psi\colon A \to X$, for a set A,

$$f \circ \phi = f \circ \psi \implies \phi = \psi.$$

It is surjective if and only if for for any functions $\alpha, \beta\colon Y \to B$, for a set B,

$$\alpha \circ f = \beta \circ f \implies \alpha = \beta.$$

As category theorists, this definition is much more appealing. It still encodes that same information as before but now we are talking exclusively in terms of functions and compositions. Our generalisations of these ideas then use exactly these definitions. The analogue of an injective function is a monomorphism, whilst that of a surjective function is an epimorphism.

Definition 1.4.1. Let $\mathbf{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathbf{C})$ and let $f\colon X \longrightarrow Y$ be a morphism.

f is called *a monomorphism*, or *monic*, if, for any $A \in \mathrm{Ob}(\mathbf{C})$ and morphisms $\phi, \psi\colon A \longrightarrow X$,

$$f \circ \phi = f \circ \psi \implies \phi = \psi.$$

f is called *an epimorphism*, or *epic*, if, for any $B \in \mathrm{Ob}(\mathbf{C})$ and morphisms $\alpha, \beta\colon Y \longrightarrow B$,

$$\alpha \circ f = \beta \circ f \implies \alpha = \beta.$$

Examples.

1. In a lot of familiar categories where the morphisms are set functions of some kind, the monomorphisms are just injective functions, whilst epimorphisms are surjective functions. This is true in **Set**, **Grp**, **Top** and others.
2. In our favourite category $\mathbf{Cov}_X$ of covering spaces over a topological space X this remains true.
3. We should not be too eager though. In **Ring**, it is not the case that epimorphisms need to be surjective. If R is an integral domain and $K(R)$ is its field of fractions, then the inclusion $R \hookrightarrow K(R)$ is an epimorphism as maps on $K(R)$ are determined by where they take R. However, if R is not a field, this map is not surjective. This map is also a monomorphism, and so is a counterexample to the intuitive assumption that morphisms that are both monic and epic must be isomorphisms. In fact, this is only true in some categories.
4. Even a slight change to our category can change what morphisms are epic and monic. In the category **HauTop** of Hausdorff topological spaces with continuous maps, epimorphisms need not be surjective. Any continuous map with a dense image is epic.

1.4.2 Final and Initial Objects

This particular definition is not about taking an a definition we have and categorising it, but instead about noticing similar objects in a few categories we are familiar with. In each of **Set**, **Top**, **Grp** and **Ring**, we have objects that are in some way canonically mapped into every other object of the category. In **Grp**, one such object is the trivial group. Given a group G, there is exactly one homomorphism $\{0\} \to G$. In **Ring**, $\mathbb{Z}$ plays a similar role: for any ring R there is exactly one ring homomorphism $\mathbb{Z} \to R$. In **Set** the object is the empty set, and in **Top** it is the empty space.

Definition 1.4.2. In a category $\mathbf{C}$, an object $A \in \mathrm{Ob}(\mathbf{C})$ is called *an initial object* if and only if for every other object $B \in \mathrm{Ob}(\mathbf{C})$ there is exactly one morphism $f\colon A \longrightarrow B$.

Remark. Initial objects are *unique up to unique isomorphism.* If A and A' are both initial objects in $\mathbf{C}$, not only is there an isomorphism $f\colon A \longrightarrow A'$ but there is only this one. This follows from the fact that all of $\mathrm{Hom}_{\mathbf{C}}(A, A)$, $\mathrm{Hom}_{\mathbf{C}}(A, A')$, $\mathrm{Hom}_{\mathbf{C}}(A', A)$ and $\mathrm{Hom}_{\mathbf{C}}(A', A')$ contain exactly one arrow each. We then may write of *the initial object* in a category to refer to any one, even if there may be more. As elsewhere, isomorphic objects are in some way equivalent.

What about the dual notion? This would be an object such that every other object has exactly one arrow to it. We are already well-aware of objects like this: there is exactly one way to map a set to a chosen singleton, a ring to the ring of one element, or to map any space to the point space. In fact, there is exactly one way to map any group to the trivial group.

Definition 1.4.3. In a category $\mathbf{C}$, an object $Z \in \mathrm{Ob}(\mathbf{C})$ is called *a terminal object* if and only if for every other object $Y \in \mathrm{Ob}(\mathbf{C})$ there is exactly one morphism $\phi\colon Y \longrightarrow Z$.

Remark. An initial object in $\mathbf{C}$ is a terminal object in $\mathbf{C}^{\mathrm{op}}$ and visa versa, so we can transfer the result from above: terminal objects are unique up to unique isomorphism.

Examples. 1. Note that a category doesn't need to have initial or terminal objects. The category **Field** of fields doesn't have either.

2. In a poset, an initial object is exactly *a least element*, an object that is less than or equal to every element of that poset. The same is true for a terminal object and a greatest element.

3. We again return to our favourite category $\mathbf{Cov}_X$. Consider the empty cover which is the one map $\varnothing \to X$. For any cover $p\colon Y \to X$, there is exactly one map $\varnothing \to Y$, and the below triangle commutes. Thus the empty cover is an initial object of $\mathbf{Cov}_X$.

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Note that there is exactly one map $\phi\colon Y \to X$ such that

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "p"'] & & X \arrow[dl, "\mathbf{1}_X"] \\ & X & \end{tikzcd}$$

commutes, namely $\phi = p$. Thus $\mathbf{1}_X\colon X \longrightarrow X$ is a final object in $\mathbf{Cov}_X$.

We can imagine a category with an initial object in some way starting there, where it branches off with an arrow to each object. Similarly a category with a final object in some way finishes there. No matter the mess of arrows amongst the objects, they all come together in exactly one way at the final object.

1.4.3 Products, Universal Properties and a Glimpse of the Future

In many of the categories of algebraic objects that we are familiar with, there is a sense of taking products. We are comfortable taking the Cartesian product of sets, the direct product of groups and the product of topological spaces. This is a perfect opportunity to exercise our new-found category theoretical language.

The key piece of information that the above examples of products have in common is that each element in the product, say $X \times Y$ has an associated projection to each of X and Y. Crucially, the element is determined entirely by where it projects. What this tells us is that choosing an element of the product $X \times Y$ is equivalent to choosing both an element of X and an element of Y; to define a function from some object $f\colon Z \to X \times Y$, it is necessary and sufficient, to give functions $f_X\colon Z \to X$ and $f_Y\colon Z \to Y$ such that $f = (f_X, f_Y)$. Calling the projections from $X \times Y$ onto X and Y π_X and π_Y respectively, that last statement says that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$. This is a property characterised entirely by morphisms and composition, which should be very appealing to us as category theorists. We should note a good opportunity for a commuting diagram:

$$\begin{tikzcd}
X & \\
X \times Y \arrow[u, "\pi_X"] \arrow[d, "\pi_Y"'] & Z \arrow[ul, "f_X"'] \arrow[l, dashed, "\exists! f"'] \arrow[dl, "f_Y"] \\
Y &
\end{tikzcd}$$

Definition 1.4.4. Let $\mathbf{C}$ be a category, and let $\{X_i\}$ be a collection of objects in $\mathbf{C}$. *The product of* $\{X_i\}$ is an object, denoted by $\prod X_i$ or $\bigtimes X_i$, along with a collection of morphisms $\{\pi_j \colon \prod X_i \longrightarrow X_j\}$ with the property that for any $Z \in \mathrm{Ob}\,(\mathbf{C})$ with a collection of morphisms $\{f_j \colon Z \longrightarrow X_j\}$ there exists a unique $f \colon Z \longrightarrow \prod X_i$ such that $\pi_j \circ f = f_j$ for all j.

When we use f, we may choose to refer to it as (f_i) or $(f_1, f_2, f_3 \ldots)$, to remember these component functions. A finite case can be illustrated with the help of a commutative diagram:

$$\begin{tikzcd}
 & X_1 & \\
 & X_2 & \\
\prod X_i \arrow[uur, "\pi_1"] \arrow[ur] \arrow[ddr, "\pi_n"'] & & Z \arrow[uul, "f_1"'] \arrow[ul] \arrow[ll, dashed, "\exists! f"] \arrow[ddl, "f_n"] \\
 & \vdots & \\
 & X_n &
\end{tikzcd} .$$

This product is unique up to unique isomorphism and the examples of products we've seen before match, which is to say that this property is sufficient to characterise all the products we have seen before. It is an example of a universal property, *the universal property of the product.* They appear in exactly the form above, a way of characterising an object and collection of morphisms by the existence of unique morphisms that make diagrams commute. It is not the only one we have seen: the definition of initial and final objects looked very similar to this. In fact, the presence of 'unique up to unique isomorphism' at the beginning of this paragraph should be jumping out as a call-back to those objects. We will see that the product is a very specific instance of a much more general construction, that of a limit, and that limits are indeed final objects in a certain category, but let's not get ahead of ourselves, there's more building to do.

The dual construction is called, as we would expect, the coproduct:

Definition 1.4.5. Let $\mathbf{C}$ be a category, and let $\{X_i\}$ be a collection of objects in $\mathbf{C}$. *The coproduct of* $\{X_i\}$ is an object, denoted by $\coprod X_i$, along with a collection of morphisms $\{q_j \colon X_j \longrightarrow \coprod X_i\}$ with the property that for any $Z \in \mathrm{Ob}\,(\mathbf{C})$ with a collection of morphisms $\{g_j \colon X_j \longrightarrow Z\}$ there exists a unique $g \colon \coprod X_i \longrightarrow Z$ such that $g \circ q_j = f_j$ for all j.

The corresponding diagram then being

$$\begin{tikzcd}
 & X_1 \arrow[ddl, "q_1"'] \arrow[ddr, "g_1"] & \\
 & X_2 \arrow[dl] \arrow[dr] & \\
\coprod X_i \arrow[rr, dashed, "\exists! g"] & & Z \\
 & \vdots & \\
 & X_n \arrow[uul, "q_n"] \arrow[uur, "g_n"'] &
\end{tikzcd} .$$

The coproduct seems to be about gluing the objects together. That the coproduct acts like an initial object, suggests that the coproduct is in a sense the 'smallest' gluing, but it also can't have too much overlap between the images of each X_i, as we can ask for two necessarily distinct arrows

$g\colon \coprod X_i \longrightarrow Z$ and $g'\colon \coprod X_i \longrightarrow Z$ that have $g_i = g_i'$ for all i but one. Common examples of coproducts then are the disjoint union of sets, or topological spaces, where no overlap is possible, or the wedge sum of pointed topological spaces, where overlap is necessary at only the one point.

Example. Let us return again to $\mathbf{Cov}_X$. The category has both finite products and finite coproducts. The coproduct of two covering spaces $p_Y\colon Y \to X$ and $p_Z\colon Z \to X$ is the map $p\colon Y \coprod Z \to X$ which takes y to $p_Y(y)$ if $y \in Y$ and $p_Z(y)$ otherwise, with the standard injections for Y and Z.

The product is a little bit more difficult. We may be tempted to consider the normal product of topological spaces, and to invent some map $p_{Y\times Z}: Y \times Z \to X$, but very rarely does this give rise to a covering space of X (although it does give a covering space of $X \times X$). The issue with the space $Y \times Z$ is that there is too much freedom, but the built in projections are handy so we may hazard a guess that we are interested instead with a subspace. Then we are looking for an appropriate subspace, which we will for the moment notate as $Y \prod Z$, and a map $p\colon Y \prod Z \to X$. We want the two projections to be morphisms in $\mathbf{Cov}_X$, so we need that

$$\begin{tikzcd} Y \arrow[dr, "p_Y"'] & Y \prod Z \arrow[l, "\pi_Y"'] \arrow[r, "\pi_Z"] \arrow[d, "p"] & Z \arrow[dl, "p_Z"] \\ & X & \end{tikzcd}.$$

commutes. Thus we must have that

$$Y \prod Z \subseteq \{(y,z) \in Y \times Z \mid p_Y(y) = p_Y(\pi_Y(y,z)) = p_Z(\pi_Z(y,z)) = p_Z(z)\}$$

and we will show in section 3.3 that $Y \prod Z$ is exactly this set with the covering defined in the diagram.

So the product in $\mathbf{Cov}_X$ of two covering spaces $p_Y: Y \to X$ and $p_Z\colon Z \to X$ is the covering space $p: \{(y,z) \in Y \times Z \mid p_Y(y) = p_Z(z)\} \to X$ given by $p(y,z) = p_Y(y)\,(= p_Z(z))$.

1.5 The Category of Separable Algebras over a Field

We are now at a point where we may introduce our next important category, the category of separable algebras over a field k. A revision of the necessary definitions and theorems can be found in appendix A.1.

Definition 1.5.1. Fix a field k. *The category of k-algebras*, denoted ${}_k\mathbf{Alg}$, has objects k-algebras and morphisms k-algebra homomorphisms.

The full subcategory of separable k-algebras will be denoted ${}_k\mathbf{SAlg}$.

Given that we are now fairly adept at understanding the basics of categories, we are going to make things more difficult for ourselves here, with the additional information that what we are doing will become relevant later. To our list of favourite categories, currently containing only $\mathbf{Cov}_X$, we are going to add the **opposite** of ${}_k\mathbf{SAlg}$, ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

This offers a good opportunity to study the constructions of section 1.4 in opposite categories.

Example. Let us investigate what final and initial object look like in this category. Since we construct the opposite category by reversing all the arrows in the original, the property that the initial object has, that it has exactly one arrow to every object, becomes that every object has exactly one arrow to it. Hence a final object in ${}_k\mathbf{SAlg}^{\mathrm{op}}$ would be an initial object of ${}_k\mathbf{SAlg}$ and an initial object in ${}_k\mathbf{SAlg}^{\mathrm{op}}$ would be a final object in ${}_k\mathbf{SAlg}$. Our question, then, becomes whether ${}_k\mathbf{SAlg}$ has final and initial objects.

Note that k is made into a k-algebra by the map $\mathbf{1}_k\colon k \to k$. For any k-algebra there is exactly one ring homomorphism $\phi\colon k \to B$ such that

$$\begin{tikzcd} & k \arrow[dl, "\mathbf{1}_k"'] \arrow[dr, "f_B"] & \\ k \arrow[rr, "\phi"] & & B \end{tikzcd}.$$

commutes. That is $\phi = f_B$, so k is the initial object in ${}_k\mathbf{Alg}$. Since it is separable (and, crucially, ${}_k\mathbf{SAlg}$ is a full subcategory of ${}_k\mathbf{Alg}$), it is the initial object in ${}_k\mathbf{SAlg}$ and thus it is the final object in ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

On the other hand, there is exactly one ring homomorphism from any k-algebra to the zero algebra $k^0 := \{0\}$ and it is also an algebra homomorphism. k^0 is the final object of ${}_k\mathbf{Alg}$. Since it is separable, it is also the final object of ${}_k\mathbf{SAlg}$ and thus it is the initial object of ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

Example. As we've seen in section 1.4.1, epimorphisms of rings are not simply surjective ring homomorphism and their classification is actually rather difficult. This behaviour transfers here and consequently it is hard for us to say anything about the monomorphisms in ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

Monomorphisms of rings are much nicer:

Claim. The monomorphisms in **Ring** are exactly the injective ring homomorphisms.

Proof. Injective functions are certainly monomorphisms. On the other hand, let $\phi\colon R \to S$ be a ring monomorphism. Let us take $r \in \operatorname{Ker}(\phi)$ and $\mathrm{ev}_r, \mathrm{ev}_0\colon R[X] \to R$ be the two homomorphisms which evaluate polynomials at r and 0 respectively. Then

$$\phi \circ \mathrm{ev}_r = \phi \circ \mathrm{ev}_0$$

so $\mathrm{ev}_r = \mathrm{ev}_0$, which finally gives us

$$r = \mathrm{ev}_r(X) = \mathrm{ev}_0(X) = 0.$$

□

Morphisms in ${}_k\mathbf{Alg}$ which are monic ring homomorphisms keep this property and in fact these are the only monomorphisms in ${}_k\mathbf{Alg}$. This can be seen in a very similar way to the above proof, replacing $R[X]$ with $k[X]$. Full subcategories inherit monomorphisms and so the injective morphisms are the monomorphisms in ${}_k\mathbf{SAlg}$. So our conclusion is that the epimorphisms in ${}_k\mathbf{SAlg}^{\mathrm{op}}$ are the morphisms corresponding to the injective morphisms in ${}_k\mathbf{SAlg}$.

Example. The last constructions we are familiar with are products and coproducts. We approach these similarly to the last examples by considering what they look like in ${}_k\mathbf{Alg}$.

In ${}_k\mathbf{Alg}$ the product of two k-algebras R and S is just the product of the underlying rings, $R \times S$ equipped with the structure map

$$\begin{aligned}(f_R, f_S)\colon k &\to R \times S\\ a &\mapsto \left(f_R\left(a\right), f_S\left(a\right)\right).\end{aligned}$$

Note that if B and C are both separable k-algebras, then by theorem A.1.5 there exist finite sets $\{B_i\}_{i=1}^t$ and $\{C_j\}_{j=1}^n$ of finite separable field extensions of k such that $B \cong \prod_{i=1}^t B_i$ and $C \cong \prod_{j=1}^n C_j$. Then $B \times C \cong \prod_{i=1}^t B_i \times \prod_{j=1}^n C_j$ so is isomorphic to a finite product of separable field extensions of k. Thus, again by theorem A.1.5, it is a separable k-algebra. Thus, again because ${}_k\mathbf{SAlg}$ is a full subcategory of ${}_k\mathbf{Alg}$, the product of B and C in this category is $B \times C$ and this is the coproduct in ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

The coproduct in ${}_k\mathbf{Alg}$ is the tensor product.

In this chapter we have laid out the basics of category theoretic language. We have outlined what a category is, and with the assistance of commutative diagrams, have begun to unite some common objects amongst the categories we have seen under powerfully general definitions. Most importantly, we have begun to think in terms of morphisms instead of elements and functions. We will continue with this in chapter 2, moving on to functors, which are how we set up correspondences between objects in categories whilst preserving commutative diagrams, and how these allow us to build limits. With all this machinery set up, we will be able to see our first major category-theoretic construction, that of Galois categories. Our prototypical examples of these will be **FinSet**, $\mathbf{Cov}_X$ and a new category that we will be using the next chapter to define, the category of separable k-Algebras for some field k.

Chapter 2

Functors, Natural Transformations and Limits

2.1 Functors

We return to categories with a singular purpose. Our investigations in the chapter 1 set us looking at the relationships arrows between objects reveal but we've consistently been 'looking inside' categories. Now, with a toolkit of definitions and examples, we can set up the machinery to look at how categories refer to one another.

Definition 2.1.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. *A (covariant) functor from* $\mathbf{C}$ *to* $\mathbf{D}$, which we shall denote as $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ consists of

1. A mapping $\begin{aligned}\mathrm{Ob}\,(\mathbf{C}) &\to \mathrm{Ob}\,(\mathbf{D})\\ A &\mapsto FA\end{aligned}$ constructing an object in $\mathbf{D}$ for every object in $\mathbf{C}$.

2. Mappings $\begin{aligned}\mathrm{Hom}_{\mathbf{C}}(A,B) &\to \mathrm{Hom}_{\mathbf{D}}(FA,FB)\\ f &\mapsto Ff\end{aligned}$ for any $A, B \in \mathrm{Ob}\,(\mathbf{C})$.

with two conditions:

(i) For any object $A \in \mathrm{Ob}\,(\mathbf{C})$, $F\mathbf{1}_A = \mathbf{1}_{FA}$.

(ii) F is *functorial.* This means that if $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are morphisms in $\mathbf{C}$, then

$$F(g \circ f) = Fg \circ Ff \tag{2.1}$$

where the composition on the left is in $\mathbf{C}$ and the composition on the right is in $\mathbf{D}$.

Equation 2.1 says that if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

commutes, then so does

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[dr, "Fh"'] & FB \arrow[d, "Fg"] \\ & FC \end{tikzcd}.$$

In fact, this is exactly the condition that if a diagram commutes in $\mathbf{C}$, then the diagram in $\mathbf{D}$ we get by applying F to all the objects and morphisms also commutes.

Definition 2.1.2. *A contravariant functor from* $\mathbf{C}$ *to* $\mathbf{D}$ is a covariant functor from $\mathbf{C}$ to $\mathbf{D}^{\mathrm{op}}$, or $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$. These definitions are equivalent and amount to all the same things as the definition above except that in the second part the mappings are $\mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{D}}(FB,FA)$ and in the second condition we take instead that $F(g \circ f) = Ff \circ Fg$. When we refer to functors, we will always be talking about the covariant ones unless mentioned. So if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

commutes in $\mathbf{C}$, then, if F is contravariant, so does

$$\begin{tikzcd} FA & FB \arrow[l, "Ff"'] \\ & FC \arrow[ul, "Fh"] \arrow[u, "Fg"'] \end{tikzcd} .$$

Examples. Constructing a functor, then, is the process of using the objects of one category to construct objects of another, in such a way that we can keep information about relationships between objects.

1. Any category has an identity functor $\mathbf{1_C}\colon \mathbf{C} \Rightarrow \mathbf{C}$ which takes every object and every morphism to itself.

2. Given two functors $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ and $G\colon \mathbf{D} \Rightarrow \mathbf{E}$, we may form their composition $GF\colon \mathbf{C} \Rightarrow \mathbf{D}$.

3. If $\mathbf{D}$ is a subcategory of $\mathbf{C}$, then there is an inclusion functor $\mathbf{D} \longrightarrow \mathbf{C}$ that takes each object and morphism to itself in the larger category.

4. A lot of familiar categories, **Grp**, **Top**, **Ring**, have objects that are sets with structure and morphisms that are set functions with structure. Then we may define functors from these to **Set**, called *forgetful functors*, which forget the structure. For example we may take a group to its set of of elements, and leave the homomorphisms as functions.

   Forgetful functors don't need to be to **Set**. We could construct a forgetful functor from $\mathbf{Cov}_X$ to **Top** that forgets that the space is covering X or from ${}_k\mathbf{Alg}$ to **Ring** which forgets the k-algebra structure of a ring.

   These are closely related to *free functors* (by a process called *adjunction*. These take sets to the free objects in codomain category that are generated by them. For a set S, the free group on S, free R-module on S for some ring R or discrete topological space with underlying set S are all examples. Maps between sets induce maps between generators in the codomain category.

5. A great deal of category theory development originated in the study of algebraic topology. In this, we are concerned with finding algebraic objects associated with spaces which give insight on their topology. This is exactly the process of setting up and studying appropriate functors $\mathbf{Top} \Rightarrow \mathbf{Grp}$ or to other algebraic categories.

   Finding the fundamental group on pointed spaces is a functor $\pi_1\colon \mathbf{Top}_* \Rightarrow \mathbf{Grp}$, taking pointed continuous functions to group homomorphisms. Homology studies particular functors from **Top** to a category of chain complexes, cohomology is similar but the functors are contravariant. Each of these also define a number of functors from **Top** to **AbGrp**.

6. An example which will be of great importance in chapter 5 is that of *Hom functors*. Let $\mathbf{C}$ be a locally small category. Then, for every $A \in \mathrm{Ob}\,(\mathbf{C})$, $\mathbf{C}$ gives us a functor

   $$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(A, -)\colon \mathrm{Ob}\,(\mathbf{C}) &\to \mathrm{Ob}\,(\mathbf{Set}) \\ B &\mapsto \mathrm{Hom}_{\mathbf{C}}(A, B) \end{aligned}$$

   which maps an arrow $f\colon B \longrightarrow C$ to

   $$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(A, f)\colon \mathrm{Hom}_{\mathbf{C}}(A, B) &\to \mathrm{Hom}_{\mathbf{C}}(A, C) \\ g &\mapsto f \circ g. \end{aligned}$$

   We may also define in the same way a contravariant functor $\mathrm{Hom}_{\mathbf{C}}(-, B)\colon \mathbf{C} \Rightarrow \mathbf{Set}$ for an object B, which would be a covariant functor *opcatC* $\Rightarrow$ **Set**.

   Both of these are functors given to us for free by $\mathbf{C}$ itself. They are vastly useful for studying the internal structure of some categories, as well as for studying other functors on $\mathbf{C}$.

7. We might want to consider a functor that picks out the solutions to sets of equations from algebraic objects. For example, I may define a functor $F\colon \mathbf{Ring} \Rightarrow \mathbf{Set}$ that takes a ring R to the set of invertible elements of R. Any ring homomorphism takes invertible elements to invertible elements, so we can just let F act on $f\colon R \longrightarrow S$ by restricting f to $R^\times$.

   We could do the same thing for picking out roots of unity, or nilpotent elements, and we don't need to be talking about solutions in a ring, but can instead generalise to work with solutions in k-algebras, for some field k. A key observation that we will formalise later, and which will fuel the invention and study of affine group schemes in chapter 5, is that in many cases this action of 'picking out solutions to a set of equations' in a k-algebra can be done instead by looking at algebra homomorphisms into it from some other k-algebra. In other words, our functor F is doing a very similar thing to some Hom functor $\mathrm{Hom}_{\mathbf{kAlg}}(A, -)$ for some k-algebra A.

8. Returning to our familiar categories, there are two functors that will be of more use in the immediate future. They involve our two favourite categories: $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}^{\mathrm{op}}$, and both go to **FinSet**. In chapter 4, we will show these are our two main examples of Galois categories, and that these functors are fundamental to this structure.

   Let X be a topological space, and let x be some point in that space. Then we define a functor

   $$F_x\colon \mathbf{Cov}_X \Rightarrow \mathbf{FinSet}.$$

   It maps the objects of $\mathbf{Cov}_X$ by

   $$(p\colon Y \to X) \mapsto p^{-1}(x).$$

   and we note that if we have a morphism of covering spaces $f\colon Y \to Z$ from $p_Y\colon Y \to X$ to $p_Z\colon Z \to X$, then $p_Y = p_Z \circ f$. So if $y \in p_Y^{-1}(x)$ then $f(y) \in p_Z^{-1}(x)$ and we can define

   $$\begin{aligned} F_x f = f\restriction_{p_Y^{-1}(x)}\colon p_Y^{-1}(x) &\to p_Z^{-1}(x) \\ y &\mapsto f(y) \end{aligned}$$

   Clearly F_x maps identity morphisms to identity morphisms, and it is functorial because for any functions $g \circ h\restriction_A = g\restriction_B \circ h\restriction_A$ if $h(A) \subset B$, so this is a functor. It may seem like an arbitrary choice, but covering spaces are fundamental in algebraic topology exactly because the way fibres of different covering spaces map between each other contains information about the shape of the base space. It is in fact this behaviour we generalise in our formalisation of Galois categories.

   On ${}_k\mathbf{SAlg}^{\mathrm{op}}$, we look for a functor ${}_k\mathbf{SAlg}^{\mathrm{op}} \longrightarrow \mathbf{FinSet}$, which is just the same a contravariant functor ${}_k\mathbf{SAlg} \longrightarrow \mathbf{FinSet}$. We are in luck: ${}_k\mathbf{SAlg}$ is a locally small category, and so we return to example 4 and find a nicely behaved contravariant functor is the Hom functor $\mathrm{Hom}_{\mathbf{kSAlg}}(-, B)$ for some separable k-algebra B. When we return to it, we will use $\mathrm{Hom}_{\mathbf{kSAlg}}(-, \bar{k})$ which is contravariant from ${}_k\mathbf{SAlg}$ to **FinSet**, because $\mathrm{Hom}_{\mathbf{kSAlg}}(B, \bar{k})$ is always finite.

The presence of identity functors and an associative way of composing them is a hint at the fact that we might want to study a category that has certain categories as objects and functors as arrows. There is a lot of structure to build there, and a lot to be said about such categories but we won't be touching on them here. However, it does motivate one definition:

Definition 2.1.3. Given two categories, $\mathbf{C}$ and $\mathbf{D}$, a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ is called *an isomorphism of categories* if it admits an inverse: a functor $G\colon \mathbf{D} \Rightarrow \mathbf{C}$ with $GF = \mathbf{1_C}\colon \mathbf{C} \Rightarrow \mathbf{C}$ and $FG = \mathbf{1_D}\colon \mathbf{D} \Rightarrow \mathbf{D}$. In this case, we say that $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic*.

Example. We noted in the example at the end of section 1.3 that the category $\mathbf{Top}_*$ of pointed topological spaces and the coslice category $\{*\}/\mathbf{Top}$ are isomorphic.

Note that this is a very strong condition. We ask that all the objects and morphisms of the two categories are in direct one-to-one correspondence with one another. In fact, this might actually be too strong for our purposes. Early on in our study of finite dimensional vector spaces on $\mathbb{R}$, we learn that any of these spaces of a given dimension n are isomorphic. This means that all the information about the spaces in this category can be discovered with reference only to the spaces $\mathbb{R}^n$. This suggests that in some way the category of finite $\mathbb{R}$-vector spaces and the category with objects $\mathbb{R}^n$ and linear maps as arrows contain the same data but they are certainly not isomorphic. We will develop a better definition, that of equivalence of categories, in section 2.4, but we'll need some new machinery first.

2.2 A New Perspective on Diagrams

It is worth remembering our examples of categories so far have been rather complicated. We can construct a category $\mathbf{C}_1$ with exactly two objects and exactly two non-equal morphisms from one to the other, which looks like $\bullet \rightrightarrows \bullet$. Perhaps we define a category $\mathbf{C}_2$ with three objects and the only non-identity arrows being exactly those in this diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \\ & \bullet \end{tikzcd}$$

that compose so as to make diagram commute. For some category $\mathbf{D}$, What do functors $F_1 \colon \mathbf{C}_1 \Rightarrow \mathbf{D}$ and $F_2 \colon \mathbf{C}_2 \Rightarrow \mathbf{D}$ look like? F_1 must pick two objects $A, B \in \mathrm{Ob}(\mathbf{D})$ and two morphisms, $f, g \in \mathrm{Hom}_{\mathbf{D}}(A, B)$. In other words, it gives us a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Note that these morphisms do not need to be equal, though they could be. On the other hand, F_2 must pick three objects in $\mathbf{D}$, and it must pick arrows between them in the arrangement of that diagram. Lastly, the corresponding diagram in $\mathbf{D}$ must commute. On the other hand, if I have a commutative diagram in $\mathbf{D}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{2.2}$$

then I can define a functor $F_2 \colon \mathbf{C} \Rightarrow \mathbf{D}$ exactly as we would expect. The fact that eq. (2.2) commutes in $\mathbf{D}$ means that F_2 is functorial. Finding a commutative diagram of this shape is exactly the same as finding a functor $F_2 \colon \mathbf{C} \Rightarrow \mathbf{D}$.

We may go further than this. When we first defined a diagram in a category, we did so as a directed multigraph where the vertexes are objects of the category and the edges are morphisms. If we take the underlying directed multigraph, we may make a category that consists of vertexes as objects, edges as arrows and then with the addition of identity arrows and arrows where ever composition is possible.

Definition 2.2.1. Given a directed multigraph Γ, *the free category generated by* Γ is the category with objects $V(\Gamma)$ and with arrows between two vertexes being any directed path between them, composed by concatenation.

and our discussion of F_1 and F_2 generalises to the statement

Theorem 2.2.2. *There is a correspondence between diagrams in a category* $\mathbf{C}$ *in the shape of* Γ *and functors from the free category generated by* Γ *to* $\mathbf{C}$.

However, we are unnecessarily restricted by the fact that these free categories must be small, and we have the additional problem that these directed multigraphs don't encode any information about composition. This motivates a revisited definition of diagrams which is in part justified by the theorem.

Definition 2.2.3. Let $\mathbf{C}$ and $\mathbf{J}$ be a categories. *A diagram of type* $\mathbf{J}$ *in* $\mathbf{C}$ is a functor $D \colon \mathbf{J} \Rightarrow \mathbf{C}$. We call $\mathbf{J}$ the *index category of* D.

This allows us to approach all the same diagrams before, but now we may also consider diagrams of large type, or diagrams that encode certain rules of composition like in F_2 above. Further improving on our approach of trying to look at objects less, statements like 'for any two objects in $\mathbf{C}$' can be converted into statements like 'for any diagram of shape $\bullet\ \bullet$'. When we introduce limits and colimits in chapter 3, of which products, coproducts and terminal objects are examples, this new definition of diagrams will be crucial.

2.3 Natural Transformations

In the same way that functors were defined because we want to understand how different categories may relate to one another, we may notice that certain functorial constructions are also related to one another. We may observe the result in algebraic topology that the first homology group of a topological space X is isomorphic to the abelianisation of the fundamental group of X. Further, when our functors turn continuous maps into homomorphisms of these groups, they do so in a way that is compatible with the abelianisation. Constructing the double dual of a vector space V comes with a way of injecting V into V^{**}, which means that the functor that constructs V^{**} is related 'in a natural' or 'canonical' way to the identity functor that just leaves V alone. Similarly, this behaves nicely with the how linear maps are dualised. These connections are valuable not just because the objects themselves map to each other nicely, but also because morphisms map nicely too. The following definition lets us formalise this 'naturality'.

Definition 2.3.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Let $G, F \colon \mathbf{C} \Rightarrow \mathbf{D}$ be functors.

A natural transformation from F to G, notated as $\alpha\colon F \Rightarrow G$, consists of a collections of morphisms in $\mathbf{D}$, indexed by the objects of $\mathbf{C}$, $\alpha_A \colon FA \longrightarrow GA$ with the following *naturality condition*:

For any $A, B \in \mathrm{Ob}(\mathbf{C})$ and any morphism $f\colon A \longrightarrow B$, the following diagram commutes in $\mathbf{D}$

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd} \tag{2.3}$$

Breaking down this definition, a natural transformation gives us a way of moving via morphisms from the image of one functor to the image of another, whilst eq. (2.3) tells us that these morphisms must behave well with how the functors transform arrows.

Examples. 1. For any functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$, then there exists an identity natural transformation, denote $\mathbf{1}_F \colon F \Rightarrow F$, whose morphisms are all identity morphisms.

2. Let us formalise our dual spaces example. First, some preliminaries:

Definition 2.3.2. Fix a field k. Then define *the category of k-vector spaces*, $\mathbf{Vect}_k$, to have k-vector spaces as its objects and linear maps as morphisms.

For a k-vector space V, *the dual of V*, notated by V^* is the set $\mathrm{Hom}_{\mathbf{Vect_k}}(V, k)$. This set is itself a k-vector space. If we have a linear map $f\colon V \longrightarrow W$ then this induces a linear map, called *the transpose of f*, $f^*\colon W^* \longrightarrow V^*$, which takes $\phi\colon W \longrightarrow k$ to $\phi \circ f\colon V \longrightarrow k$.

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, dashed, "f^*(\phi)"'] & & W \arrow[dl, "\phi"] \\ & k & \end{tikzcd}$$

Since

$$(f_1 \circ f_2)^*(\phi) = \phi \circ (f_1 \circ f_2) = (\phi \circ f_1) \circ f_2 = {f_2}^*(\phi \circ f_1) = {f_2}^* \circ {f_1}^*(\phi)$$

this is functorial so we have a functor $(-)^*\colon \mathbf{Vect}_k \Rightarrow \mathbf{Vect}_k^{\mathrm{op}}$ (or indeed $\mathbf{Vect}_k^{\mathrm{op}} \Rightarrow \mathbf{Vect}_k$)

We can meaningfully define $(-)^{**}\colon \mathbf{Vect}_k \Rightarrow \mathbf{Vect}_k$ as the composition of $(-)^*$ with itself.

Definition 2.3.3. The functor $(-)^{**}\colon \mathbf{Vect}_k \Rightarrow \mathbf{Vect}_k$ takes a vector space V to the dual of its dual. Let $f\colon V \longrightarrow W$ be a linear map. Then $f^{**} := (f^*)^*\colon V^{**} \longrightarrow W^{**}$.

What does f^{**} look like? An element of V^{**} is a linear map $\phi\colon V^* \longrightarrow k$, and $f^{**}(\phi) = \phi \circ f^*$. This gives an element of W^{**}, a linear map $W^* \longrightarrow k$. Explicitly, given a linear $\mu\colon W \longrightarrow k$,

$$f^{**}(\phi)(\mu) = \phi \circ f^*(\mu) = \phi(\mu \circ f).$$

To define a natural transformation between $\mathbf{1}_{\mathbf{Vect}_k}$ and $(-)^{**}$, we see from definition 2.3.1 that for any k-vector space V we need a linear map $\alpha_V\colon V \longrightarrow V^{**}$. This whole example is motivated by the fact that there seems to be an obvious way to do this.

An important class of elements in V^{**} are the evaluation maps. If $v \in V$ then the map $\mathrm{eval}_v\colon V^* \longrightarrow k$ takes $\mu\colon V \longrightarrow k$ to its value at v, $\mu(v)$. Then we define the map

$$\begin{aligned} \alpha_V \colon V &\to V^{**} \\ v &\mapsto \mathrm{eval}_v. \end{aligned}$$

If V is finite dimensional, this is an isomorphism, but it doesn't need to be in the infinite dimensional case.

So our last step then is to see if this is a natural choice. This comes down to checking if this diagram commutes for any two vector spaces V and W and any linear map $f\colon V \longrightarrow W$,

$$\begin{tikzcd} \mathbf{1}_{\mathbf{Vect}_k}(V) = V \arrow[r, "\alpha_V"] \arrow[d, "\mathbf{1}_{\mathbf{Vect}_k}(f)=f"'] & V^{**} \arrow[d, "f^{**}"] \\ \mathbf{1}_{\mathbf{Vect}_k}(W) = W \arrow[r, "\alpha_W"'] & W^{**}. \end{tikzcd}$$

So let us take an element $v \in V$ and follow it around both sides of the diagram. Firstly the top and right edges:

$$f^{**} \circ \alpha_V(v) = f^{**}(\mathrm{eval}_v)$$

and then the left and bottom edges:

$$\alpha_W \circ f(v) = \mathrm{eval}_{f(v)}.$$

Given $\mu \in W^*$, we have

$$\begin{aligned} f^{**}(\mathrm{eval}_v)(\mu) &= \mathrm{eval}_v(\mu \circ f) \\ &= \mu \circ f(v) \\ &= \mu(f(v)) \\ &= \mathrm{eval}_{f(v)}(\mu) \end{aligned}$$

so

$$f^{**}(\mathrm{eval}_v) = \mathrm{eval}_{f(v)}$$

and the diagram commutes.

3. Given functors $F, G, H\colon \mathbf{C} \Rightarrow \mathbf{D}$ and natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, we may make $\beta\alpha\colon F \Rightarrow G$ by $(\beta\alpha)_A = \beta_A \circ \alpha_A$. This composition is associative.

We won't elaborate on it here, but as in the case of functors the presence of identities and composition hints at the fact that the category of functors $\mathbf{C} \Rightarrow \mathbf{D}$ with arrows given by natural transformations is well defined.

Definition 2.3.4. A natural transformation, for $F, G\colon \mathbf{C} \Rightarrow \mathbf{D}$, $\alpha\colon F \Rightarrow G$ is called *a natural isomorphism* if there exists an inverse natural transformation $\beta\colon G \Rightarrow F$. This is exactly when each α_A is an isomorphism.

Example. If we only consider the category of finite vector spaces over a field k then the natural transformation we described above between the identity functor and $(-)^{**}$ is a natural isomorphism. This allows us to say that a finite k-vector space is naturally isomorphic to its double-dual.

2.4 Equivalences

Back with definition 2.1.3 we defined an isomorphism of categories but we noted that the conditions required were particularly strong. Let us use our category theory toolbox to construct a definition that might suit us better. We certainly are looking for a correspondence of some kind, so given two categories $\mathbf{C}$ and $\mathbf{D}$ we would like some conditions on a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ (and perhaps a second functor $G\colon \mathbf{D} \Rightarrow \mathbf{C}$). The main problem in our example with the category of $\mathbb{R}^n$s and linear maps and the category of $\mathbf{Vect}_{\mathbb{R}}$ was that the same information was contained in any class of isomorphic objects. Therefore, we might not care if our functor is surjective on objects, we might just want to make sure that it touches on every isomorphism class.

Definition 2.4.1. A functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ is called *essentially surjective* if and only if for any $B \in \mathrm{Ob}(\mathbf{D})$ there is an $A \in \mathrm{Ob}(\mathbf{C})$ such that $FA \cong B$.

Relaxing the situation for objects means that we now can't expect to map to every morphism. However, it doesn't seem right to consider two categories to be equivalent if one has 'more' morphisms than the other. These may contain extra information, since arrows between two objects have no sense of being isomorphic to one another, so we do need some sort of surjectivity on morphisms.

Definition 2.4.2. A functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ is called *full* if and only if for any $A, B \in \mathrm{Ob}\,(\mathbf{C})$ and any $h \in \mathrm{Hom}_{\mathbf{D}}(FA, FB)$, there is some $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ such that $Ff = h$.

In section 1.3 we defined a full subcategory. We can see now that a full subcategory is exactly a subcategory for which the inclusion functor is full.

The last condition we need is similarly some sort of injectivity on morphisms. In a similar way to with fullness, we don't want two morphisms in our first category to contain the same information as one in the second.

Definition 2.4.3. A functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ is called *faithful* if and only if for any $A, B \in \mathrm{Ob}\,(\mathbf{C})$ and any $f, g\colon A \longrightarrow B$, $Ff = Fg$ implies that $f = g$.

In locally small categories, fullness and faithfulness are exactly surjectivity and injectivity on Hom sets.

With these conditions we have enough information to define an equivalence of categories.

Definition 2.4.4. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We say $\mathbf{C}$ *is equivalent to* $\mathbf{D}$, written as $\mathbf{C} \simeq \mathbf{D}$, if there exists a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ that is essentially surjective, full and faithful.

There is something unsatisfying about this definition. In all the categories we've dealt with, when we defined isomorphisms, the closest things we have to equivalences, we surmised this idea of 'containing the same information/structure' concisely with reference to inverses, whereas here we are left with a list of conditions. It isn't even immediately obvious that $\mathbf{C} \simeq \mathbf{D}$ implies that $\mathbf{D} \simeq \mathbf{C}$, a property that we certainly want.

So let us try a different approach. Instead of a single functor let us take two, $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ and $G\colon \mathbf{D} \Rightarrow \mathbf{C}$. If we were requiring that these categories were isomorphic, we would want these functors such that $GF = \mathbf{1_C}$ and $FG = \mathbf{1_D}$. We want to weaken this in such a way that we aren't actually losing any information. We do not need these compositions to be equal to the identities, we only need them to contain the same information and our discussion about natural transformations has given us exactly the vocabulary to define this.

Definition 2.4.5. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. *An equivalence between* $\mathbf{C}$ *and* $\mathbf{D}$ is a quadruple (F, G, α, β) where $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ and $G\colon \mathbf{D} \Rightarrow \mathbf{C}$ are functors and $\alpha\colon \mathbf{1_C} \Rightarrow GF$ and $\beta\colon \mathbf{1_D} \Rightarrow FG$ are natural isomorphisms. F and G are said to be *weak inverses.* If we have such an equivalence, we write that $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, written $\mathbf{C} \simeq \mathbf{D}$.

Proposition 2.4.6. *Given a equivalence of categories between* $\mathbf{C}$ *and* $\mathbf{D}$*,* (F, G, α, β)*,* F *is full, faithful and essentially surjective. On the other hand, if we have a full, faithful and essentially surjective* $F\colon \mathbf{C} \Rightarrow \mathbf{D}$*, we can construct an equivalence* (F, G, α, β)*, assuming a sufficiently strong axiom of choice.*

Proof. We only prove the first statement, since it is enough to show that our new definition is at least as strong as our first one. The proof of the second can be found in [2].

Let us take an equivalence (F, G, α, β).

Take $A, B \in \mathbf{C}$ and let $f, g\colon A \longrightarrow B$ with $Ff = Fg$. Then $GFf = GFg$. By the naturality of α, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"', "\cong"] & B \arrow[d, "\alpha_B"', "\cong"] \\ GFA \arrow[r, "GFf"] & GFB \end{tikzcd}$$

and, since α_A is an isomorphism, can see that

$$f = \alpha_B^{-1} \circ GFf \circ \alpha_A = \alpha_B^{-1} \circ GFg \circ \alpha_A = g$$

so F is faithful. By the same method, G is also faithful.

Now let $h\colon FA \longrightarrow FB$ and define $f = \alpha_B^{-1} \circ Gh \circ \alpha_A \in \mathrm{Hom}_{\mathbf{C}}(A, B)$.

$$\begin{tikzcd} A \arrow[r, dashed, "f"] \arrow[d, "\alpha_A"', "\cong"] & B \arrow[d, "\alpha_B"', "\cong"] \\ GFA \arrow[r, "Gh"] & GFB \end{tikzcd}$$

But we note then that this means $Gh = \alpha_B \circ f \circ \alpha_A^{-1} = GFf$, and G is faithful, so $h = Ff$ and we have that F is full.

Finally, given $B \in \mathrm{Ob}\,(\mathbf{D})$, we have the isomorphism $\beta_B \colon B \xrightarrow{\sim} FGB$ so F is essentially surjective. $\square$

Examples. 1. Let's take k to be a field and consider the category ${}_{\mathrm{fin}}\mathbf{Vect}_k$ of finite k-vector spaces and its opposite ${}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}}$. We already have functors $(-)^* \colon {}_{\mathrm{fin}}\mathbf{Vect}_k \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}}$ and $(-)^* \colon {}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}} \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k$ (with a slight abuse of notation) and in the examples above we defined a natural transformation between $\mathbf{1}_{{}_{\mathrm{fin}}\mathbf{Vect}_k}$ and $(-)^{**} \colon {}_{\mathrm{fin}}\mathbf{Vect}_k \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k$. In the same way, may find one between $\mathbf{1}_{{}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}}}$ and $(-)^{**} \colon {}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}} \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}}$. Thus we find ${}_{\mathrm{fin}}\mathbf{Vect}_k \simeq {}_{\mathrm{fin}}\mathbf{Vect}_k^{\mathrm{op}}$.

2. To formalise our motivating example of $\mathbb{R}^n$, we'll work in the more general environment of ${}_{\mathrm{fin}}\mathbf{Vect}_k$ again. Let us define a new category, which we will call $\mathbf{Mat}_k$. We let $\mathrm{Ob}(\mathbf{Mat}_k) = \mathbb{N}_0$ and define $\mathrm{Hom}_{\mathbf{Mat}_k}(n, m)$ to be the set of all $m \times n$ matrices with coefficients in k. An arrow with domain n is just a matrix with n columns, and an arrow with codomain l is just a matrix with l rows, and so we can define composition of arrows as multiplication of the matrices.

   For example, here is a diagram in $\mathbf{Mat}_{\mathbb{R}}$

$$\begin{tikzcd}
1 \arrow[loop left, "(\pi)"] \arrow[r, bend left, "\begin{pmatrix} 7 & 0 \end{pmatrix}"] \arrow[rr, bend right, "\begin{pmatrix} 7e & 0 & 14 & 7 \end{pmatrix}"'] & 2 \arrow[r, bend left, "\begin{pmatrix} e & 0 & 2 & 1 \\ 3 & 1 & 1 & 19 \end{pmatrix}"] \cdots & 4 \arrow[loop right, "\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"]
\end{tikzcd}$$

   where the two blue matrices have been composed to make the red one.

   Our original statement was inspired by the fact that all finite-dimensional $\mathbb{R}$-vector spaces are isomorphic to $\mathbb{R}^n$ for some n. In fact, we have the stronger result that every finite-dimensional k-vector space is isomorphic to k^n for some n.

   Proposition 2.4.7. *Let k be a field. Then the category $\mathbf{Mat}_k$ is equivalent to any category ${}_{\mathrm{fin}}\mathbf{Vect}_k^B$ of finite dimensional k-vector spaces where an ordered basis B_V has been chosen for each space V.*

   Proof. Let us define a functor $F \colon \mathbf{Mat}_k \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k^B$ which takes n to k^n and which takes a matrix $A \in \mathrm{Hom}_{\mathbf{Mat}_k}(n, m)$ to the linear map $T_A \colon k^n \to k^m$ which is given by A in the bases B_{k^n} and B_{k^m}. The one-to-one correspondence between matrices and linear maps gives that F is full and faithful, which the isomorphisms between n-dimensional spaces and k^n given that it is essentially surjective.

   If we were to use our second definition of an equivalence, we could define $G \colon {}_{\mathrm{fin}}\mathbf{Vect}_k^B \Rightarrow \mathbf{Mat}_k$ which takes a space V to $\dim(V) \in \mathbb{N}$ and any linear map map between spaces to the corresponding matrix in their bases. Then we note that $GF \colon \mathbf{Mat}_k \Rightarrow \mathbf{Mat}_k = \mathbf{1}_{\mathbf{Mat}_k}$ and that $FG \colon {}_{\mathrm{fin}}\mathbf{Vect}_k^B \Rightarrow {}_{\mathrm{fin}}\mathbf{Vect}_k^B$ takes an n-dimensional vector space V to k^n and a linear map from an n-dimensional space to an m-dimensional space to the map $k^n \to k^m$ given by the same matrix.

   So $\alpha \colon \mathbf{1}_{\mathbf{Mat}_k} \Rightarrow GF$ and $\beta \colon \mathbf{1}_{{}_{\mathrm{fin}}\mathbf{Vect}_k^B} \Rightarrow FG$ are just the identity natural transformation, and the linear isomorphisms $\beta_V \colon V \to k^n$ which takes the basis B_V to the basis B_{k^n}. $\square$

   So all that stops the study of finite dimensional k-vector spaces from being the study of matrices in k is the freedom of choice of basis.

Chapter 3

Limits and Colimits

We have now seen a number of category-theoretic constructions.They have helped us to notice similarities between categories that go beyond similarities of their objects. The presence of initial objects or products are important statements about the shape of the category and how its objects relate to each other. Their definitions were informed by a long list of examples in the familiar categories of algebraic objects. Each definition, then, looked like ones that we had seen many times before. It was not too difficult to wrap our heads around category theoretic products because they were defined in much the same way as direct products, ring products or Cartesian products. However, as category theory was developed to do, by stepping away from this for a moment, and dealing in some definitions and ideas that are a little more unfamiliar, we can begin to connect ideas which seem only tenuously similar.

Back in section 1.4.3, in our discussion of products and coproducts, we noticed that the presence of "unique morphisms" or objects that were "unique up to unique isomorphism" repeated the language of initial and final objects. We will find, in fact, they *are* initial and final objects, just not in the categories where they normally live. They are not the only such constructions. Once we develop the definitions necessary to make these categories, we will find that all sorts of mathematical objects can be seen as either initial or final objects in appropriate categories. Wherever we might find ourselves "gluing" objects in categories, like coproducts but also for quotients of groups, rings and vector spaces, or when we look for sub-things satisfying some properties, such as the preimage of a set function, the equaliser of two continuous maps, or a level set of a smooth map on a manifold, we are using colimits and limits respectively [1].

3.1 Cocones and Colimits

Let us focus, for a moment, on the "gluing" case. As we saw in section 2.2, we may use functors from a chosen category $\mathbf{C}$ to another $\mathbf{D}$ as diagrams in the shape of $\mathbf{C}$ in $\mathbf{D}$. This is a strong generalisation of indexing sets with other sets, because not only do functors pick out objects, but they pick morphisms too and further give us constraints on the composition of these morphisms. So approaching colimits, particularly given our perspective about "gluing", we begin with a functor, which allows us to choose objects to glue and, crucially, morphisms along which we will be gluing.

Definition 3.1.1. Let $\mathbf{I}$ be an index category and let $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram. *A cocone for* D *consists of an object* $C \in \mathrm{Ob}(\mathbf{C})$ *and for each* $i \in \mathrm{Ob}(\mathbf{I})$ *a morphism* $c_i\colon D_i \longrightarrow C$*. Further, if* $\alpha\colon i \longrightarrow j \in \mathrm{Hom}_{\mathbf{I}}(i,j)$*, then* $c_j \circ D(\alpha) = c_i$*.*

Let us consider what this means with help of some examples:

Examples.

1. Let $\mathbf{I}$ be the category with two objects and no non-identity morphisms. Then a diagram $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ just picks two objects $D_1, D_2 \in \mathrm{Ob}(\mathbf{C})$ and a cocone of D consists of an object $C \in \mathrm{Ob}(\mathbf{C})$ and choice of morphisms $c_i\colon D_i \longrightarrow C$ for $i = 1,2$. Since there are no non-identity morphisms in $\mathbf{I}$, so our last condition is redundant.

2. A more illustrative example, perhaps, is to consider the index category $\mathbf{I}$ which looks like $\bullet \longleftarrow \bullet \longrightarrow \bullet$. A diagram $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ then, consists of a choice of three objects and

[1]Pedagogically, this perspective is explored excellently in [1]

two morphisms in that arrangement:

$$\begin{tikzcd} D_1 \arrow[r, "f_2"] \arrow[d, "f_3"'] & D_2 \\ D_3 & \end{tikzcd}$$

So to determine a cocone of this diagram, we need a $C \in \mathrm{Ob}(\mathbf{C})$ and morphisms for each D_i.

$$\begin{tikzcd} D_1 \arrow[r, "f_2"] \arrow[d, "f_3"'] \arrow[dr, "c_1"] & D_2 \arrow[d, "c_2"] \\ D_3 \arrow[r, "c_3"] & C \end{tikzcd}$$

They also need to fulfil our last condition, which is exactly that if we start at an object D_i, it is the same to pass directly to C via c_i as to move to another D_j and then along c_j. Since the only non-identity morphisms are the f_is, we only need to worry that

$$c_2 \circ f_2 = c_1$$

and

$$c_3 \circ f_3 = c_1.$$

In other words, we need the following diagram to commute

$$\begin{tikzcd} D_1 \arrow[r, "f_2"] \arrow[d, "f_3"'] & D_2 \arrow[d, "c_3"] \\ D_3 \arrow[r, "c_2"] & C \end{tikzcd}.$$

Cocones of D then are exactly these commutative diagrams.

3. If $\mathbf{I}$ is the category $\bullet \rightrightarrows \bullet$, then diagrams look like $D_1 \overset{f}{\underset{g}{\rightrightarrows}} D_2$ and cocones are simply maps $c\colon D_2 \longrightarrow C$ such that $c \circ f = c \circ g$.

Remark. The condition in the definition of a cocone seems rather restrictive. We can imagine the more morphisms and objects we add to $\mathbf{I}$, the hard it would be to find cocones. This intuition is not quite right, as we will see in section 4.1, but there is a simple case where it fails. If the category $\mathbf{C}$ has a final object Z, then for a diagram $D\colon \mathbf{I} \Rightarrow \mathbf{C}$, Z and the unique morphism $D_i \longrightarrow Z$ for each D_i are a cocone of D

Since we are leaning into the morphism-focused approach, the only way we can evaluate something like "gluing" is by using morphisms to give some sense of inclusion. Each of the c_i morphisms above, is a way of putting D_i into C, and the condition about composition gives us information about where these copies of D_i and D_j must overlap, or be glued. However, we may in this way lay out what must be included, and what must be glued, but there's no constraint on how much gluing we do. The third example above is the most extreme form of this: everything maps into the final object, and it is the case of gluing everything to everything. So what we are looking for is not just a cocone, but in some sense a minimal cocone, which glues exactly what we want and no more. Again, we are reminded of the language of initial objects.

Definition 3.1.2. Let $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram, and let $(A, (a_i))$ and $(B, (b_i))$ be cocones of D. *A morphism of cocones from A to B* is a morphism $e\colon A \longrightarrow B$ such that $e \circ a_i = b_i$ for every $i \in \mathbf{I}$.

So we return to our examples:

Examples. 1. Given two cones A and B, We are looking for a morphism e such that the following diagram commutes:

$$\begin{tikzcd} D_1 \arrow[d, "a_1"'] \arrow[dr, "b_1"] & \\ A \arrow[r, "e"] & B \\ D_2 \arrow[u, "a_2"] \arrow[ur, "b_2"'] & \end{tikzcd}$$

2. Again, let A and B, be cones. Then we need e such that this diagram commutes:

$$\begin{tikzcd}
& D_2 \arrow[dr, "a_2"] \arrow[drr, bend left, "b_2"] & & \\
D_1 \arrow[ur, "f_2"] \arrow[dr, "f_3"'] & & A \arrow[r, "e"] & B \\
& D_3 \arrow[ur, "a_3"'] \arrow[urr, bend right, "b_3"'] & &
\end{tikzcd}$$

3. In this case, we want e as below:

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_2 \arrow[r, "b"] \arrow[dr, "a"'] & B \\
& & A \arrow[u, "e"']
\end{tikzcd}$$

With morphisms defined, and a simple verification of the axioms, we now have *the categories of cocones over a diagram* $D\colon \mathbf{I} \Rightarrow \mathbf{C}$.

If $\mathbf{C}$ has a final object, then this too forms a final object in the category of cocones of any diagram into $\mathbf{C}$. We are more interested in the existence of initial objects.

Definition 3.1.3. Let $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram. If it exists, the initial object in the category of cocones of D is called *the colimit of* D and is denoted by

$$\varinjlim D_i.$$

We say it is a *finite colimit* if $\mathbf{I}$ is a finite category.

As before, it is an abuse of notation to talk of *the* initial object, or *the* colimit, but we know that all initial objects in a category are unique up to unique isomorphism so if we keep this in mind we are safe. Let us consider our examples above.

Examples. 1. The initial object in the category of cocones on the diagram D, $(\varinjlim D_i, \imath_i)$ would be defined entirely by the property that if (A, a_i) was another cocone, there would be a unique morphism of cocones $e\colon \varinjlim D_i \longrightarrow A$. In other words, the existence of a unique e such that this diagram commutes:

$$\begin{tikzcd}
D_1 \arrow[d, "\imath_1"'] \arrow[dr, "a_1"] & \\
\varinjlim D_i \arrow[r, dashed, "e"] & B \\
D_2 \arrow[u, "\imath_2"] \arrow[ur, "a_2"'] &
\end{tikzcd}$$

but this is exactly the universal property of the coproduct and in a category with coproducts, any object satisfying this property is uniquely isomorphic to any other. Thus we may alternatively define the coproduct as the colimit of this diagram. Then we can expand that to finite coproducts as the colimit of diagrams in the form of the category of n discrete objects. This also seems to suggest we have the right definitions, as coproducts are the minimal way of gluing two objects.

2. For our second example, let us work in **Set**. We would like to find the colimit of a diagram $D_3 \xleftarrow{f_3} D_1 \xrightarrow{f_2} D_2$. If our definition is close enough to what we intended, then we would hope $\varinjlim D_i$ resembles a gluing of D_1, D_2 and D_3 along some well chosen subset.

Claim. The equivalence classes of the disjoint union $D_2 \coprod D_3$ under the equivalence relation generated by

$$\imath_2 \circ f_2(x) \sim \imath_3 \circ f_3(x),$$

where $\imath_i$ is the inclusion into the coproduct, is the colimit of the diagram (again with the natural inclusions $\jmath_i\colon D_i \longrightarrow D_2 \coprod D_3/\sim$).

Proof. It is clear that this is a cocone of the diagram, since we define the equivalence classes to satisfy exactly the commutativity requirement.

We must show it is initial in the category of cocones. Let

$$\begin{tikzcd} D_1 \arrow[r, "f_2"] \arrow[d, "f_3"'] & D_2 \arrow[d, "c_3"] \\ D_3 \arrow[r, "c_2"] & C \end{tikzcd} .$$

be another cocone.

By the universal property of the coproduct, we have unique maps $e: D_2 \coprod D_3 \longrightarrow C$ and $f: D_2 \coprod D_3 \longrightarrow D_2 \coprod D_3/\sim$. Also we have that for all $z \in D_1$, $c_3(f_3(z)) = c_2(f_2(z))$ so in particular e factors through f to a map $D_2 \coprod D_3/\sim \longrightarrow C$ which takes the equivalence class $[a]$ to $c_i(a)$ for $a \in D_i$. It is easy to show that all such functions must do this and so it is unique. □

Then in a similar way to what we expected, the maps from D_1 can be seen as identifying the elements to glue together. The claim is also true in **Top** and lends credence to this geometric interpretation of gluing.

The limit of such a diagram is called *the pushout of arrows* $f_i: D_1 \longrightarrow D_i$ and is normally denoted by $D_2 \coprod_{D_1} D_3$ to represent this idea of gluing D_2 and D_3 along the image of D_1.

3. Colimits of diagrams of the form $D_1 \underset{g}{\overset{f}{\rightrightarrows}} D_2$ are an important case of colimits called *the coequaliser of f and g*. Like in the previous example, in **Set**, the coequaliser is a quotient, but we are now gluing D_2 to itself. It is then $D_2/\sim$ where $\sim$ is the equivalence relation generated by the relations $f(x) \sim g(x)$ for all $x \in D_1$.

3.2 Cones and Limits

Let us consider the dual notions to those in the last section. Though they do not admit as simple an interpretation as the gluing of colimits, we are familiar with a number of examples. We will see that limits are a particular abstraction of finding solution sets of equations and this is where the "sub-thing" feel of many limits comes from.

Definition 3.2.1. Let $\mathbf{I}$ be an index category and let $D: \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram. *A cone for D* consists of an object $C \in \mathrm{Ob}(\mathbf{C})$ and for each $i \in \mathrm{Ob}(\mathbf{I})$ a morphism $c_j: C \longrightarrow D_i$. Further, if $\alpha: i \longrightarrow j \in \mathrm{Hom}_{\mathbf{I}}(i,j)$, then $D(\alpha) \circ c_i = c_j$.

Taking the same examples as before:

Examples. 1. Let $\mathbf{I}$ be the category with two objects and no non-identity morphisms. Cones of a diagram $D: \mathbf{I} \Rightarrow \mathbf{C}$ which picks the two objects $D_1, D_2 \in \mathrm{Ob}(\mathbf{C})$, is just a choice of object $C \in \mathrm{Ob}(\mathbf{C})$ and any morphisms $c_i: C \longrightarrow D_i$.

2. Let $\mathbf{I}$ be the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ then in a similar way as for the cocones, cones are choices of $C \in \mathrm{Ob}(\mathbf{C})$ and $c_i: C \longrightarrow D_i$ such that

$$\begin{tikzcd} C \arrow[r, "c_2"] \arrow[d, "c_3"] & D_2 \arrow[d, "f_2"] \\ D_3 \arrow[r, "f_3"] & D_1 \end{tikzcd}$$

commutes.

3. If $\mathbf{I}$ is the category $\bullet \rightrightarrows \bullet$, then for the diagram $D_1 \underset{g}{\overset{f}{\rightrightarrows}} D_2$, a cones is a maps $c: C \longrightarrow D_1$ such that $f \circ c = g \circ c$.

As we'd expect, these are the dual diagrams of our examples of cocones.

Remark. Again dualising, if $\mathbf{C}$ has an initial object $A \in \mathrm{Ob}(\mathbf{C})$, then there is a trivial cone for any diagram consisting of A and the unique maps $c_i: A \longrightarrow D_i$.

Cones gives us some way of putting our object into each of those in the diagram in a way that is compatible with the morphisms in the diagram. If we are working in a concrete category, like **Set**, then we can imagine that each element of $x \in C$ contains a component element for each D_i that must obey the composition of morphisms given in the diagram. This matches the idea of setting up systems of equations without have to "look inside" the objects. For example, we will see that for functions of sets $f, g\colon A \to B$, the set $\{x \in A \mid f(x) = g(x)\}$ is a limit of a particular diagram, since we looking to choose elements of A which obey a certain relation. We are again looking for the maximal such object, since we would like to encapsulate all the solutions.

Definition 3.2.2. Let $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram, and let $(A, (a_i))$ and $(B, (b_i))$ be cones of D. A *morphism of cones from A to B* is a morphism $e\colon B \longrightarrow A$ such that $a_i \circ e = b_i$ for every $i \in \mathbf{I}$.

Returning to our examples:

Examples. 1. Letting A and B be cones over the category of two discrete objects, we are looking for a morphism e such that the following diagram commutes:

$$\begin{tikzcd}
& D_1 \\
B \arrow[ur, "b_1"] \arrow[r, "e"] \arrow[dr, "b_2"'] & A \arrow[u, "a_1"'] \arrow[d, "a_2"] \\
& D_2
\end{tikzcd}$$

2. For our second diagram, morphisms of cones give the diagram

$$\begin{tikzcd}
& & D_2 \arrow[dr, "f_2"] & \\
B \arrow[r, "e"] \arrow[urr, bend left, "b_2"] \arrow[drr, bend right, "b_3"'] & A \arrow[ur, "a_2"] \arrow[dr, "a_3"'] & & D_1 \\
& & D_3 \arrow[ur, "f_3"'] &
\end{tikzcd}$$

3. For the last diagram, we need e with

$$\begin{tikzcd}
A \arrow[r, "a"] & D_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_2 \\
B \arrow[u, "e"] \arrow[ur, "b"] & &
\end{tikzcd}$$

With morphisms defined, we have *the category of cones of a diagram $D\colon \mathbf{I} \longrightarrow \mathbf{C}$*.

Definition 3.2.3. Let $D\colon \mathbf{I} \Rightarrow \mathbf{C}$ be a diagram. If it exists, the terminal object in the category of cones of D is called *the limit of D* and is denoted by

$$\varprojlim D_i.$$

We say it is a *finite limit* if $\mathbf{I}$ is a finite category.

We will again talk of *the* limit, since all limits of D are unique up to unique isomorphism.

Examples. 1. By the same reasoning as for the coproduct, the limit of our first example is the product $D_1 \times D_2$.

2. Let us work in **Set** again to wrap our heads around this second example. Our diagram consists of three sets D_1, D_2, D_3 and maps $f_i\colon D_i \longrightarrow D_1$ for $i = 1, 2$

   Claim. The limit of this diagram is given by the set

$$D_2 \times_{D_1} D_3 := \{(x, y) \in D_2 \times D_3 \mid f_2(x) = f_3(x)\}$$

   equipped with the projections onto D_2 and D_3.

Proof. Simply by following elements along arrows, we can see this is a cone. If

$$\begin{tikzcd} C \arrow[r, "c_2"] \arrow[d, "c_3"] & D_2 \arrow[d, "f_2"] \\ D_3 \arrow[r, "f_3"] & D_1 \end{tikzcd}$$

is another cone, then there is a unique morphism of cones $C \longrightarrow D_2 \times_{D_1} D_3$ which takes $z \in C$ to $(c_2(z), c_3(z))$. ☐

This tells us that we have the right idea considering limits as finding solutions to systems of equations. The limit of such a diagram is called *the pullback* or *fibre product of arrows* $f_i \colon D_i \longrightarrow D_1$. We have the same result again in **Top**.

3. The dual of a coequaliser of arrows $f, g \colon D_1 \longrightarrow D_2$ is, unsurprisingly, called *the equaliser of f and g*. As we'd hope given our intuition about solution sets, in all of **Set**, **Top** and **Grp** this is $\{x \in D_1 \mid f(x) = g(x)\}$.

Let's note that even the most basic constructions like this can give non-trivial results.

Proposition 3.2.4. *Let* **C** *be a category and let $f, g \colon X \longrightarrow Y$ be morphisms. Then, if it exists, the equaliser $e \colon E \longrightarrow X$ is a monomorphism.*

Proof. Let $\phi, \varphi \colon Z \longrightarrow E$ be morphisms. Then $e \circ \phi$ and $e \circ \psi$ both equalise f and g. The fact that the equaliser is a limit says that there is a unique morphism $\psi \colon Z \longrightarrow E$ such that $e \circ \psi = e \circ \phi$ (in other words, $\psi = \phi$) and so if $e \circ \phi = e \circ \varphi$, we must have that $\phi = \varphi$. ☐

3.3 Fibre Products in Familiar Categories

Now that we have sense of some of these constructions, let us look at the specific example of fibre products in our favourite categories, $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}^{\mathrm{op}}$.

Theorem 3.3.1. *Let X be a topological space. Let p_Y, p_Z, p_W be covering spaces and let $f \colon Y \longrightarrow Z$ and $g \colon W \longrightarrow Z$ covering space morphisms. Then the fibre product of f and g in $\mathbf{Cov}_X$ exists and is the covering space $p \colon Y \times_Z W \to X$ given by $p(y, w) = p_Y(y) = p_W(w)$, equipped with the projections π_Y and π_W, where $Y \times_Z W$ is the fibre product in* **Top**.

Proof. We need to show that the covering space we defined is a cone. By the definition of the fibre product in **Top**, we already have that the diagram

$$\begin{tikzcd} Y \times_Z W \arrow[r, "\pi_Y"] \arrow[d, "\pi_W"] & Y \arrow[d, "f"] \\ W \arrow[r, "g"] & Z \end{tikzcd}$$

commutes in **Top** and so we need only that it is indeed a covering space and that all the continuous maps are also morphisms in $\mathbf{Cov}_X$ for this to be a cone. We already know f and g are covering space morphisms, so we need that

$$\begin{tikzcd} Y \arrow[dr, "p_Y"'] & Y \times_Z W \arrow[l, "\pi_Y"'] \arrow[r, "\pi_W"] \arrow[d, "p"] & W \arrow[dl, "p_W"] \\ & X & \end{tikzcd}$$

commutes. The left triangle is the definition we have given for p, so we only need that on $Y \times_Z W$ we have $p = p_W \circ \pi_W$. But

$$p(x, w) = p_Y(y) = p_Z(f(y)) = p_Z(g(w)) = p_W(w)$$

since every $(y, w) \in Y \times_Z W$ has $f(y) = g(w)$.

To show it is a covering space, we use this lemma from [10]:

Lemma 3.3.2. *Let X be a topological space and $q_i\colon Y_i \to X$ covering spaces for $i = 1, 2$. Let $\phi\colon Y_1 \longrightarrow Y_2$ be a morphism of covering spaces. Then for any $x \in X$ there is a neighbourhood U, finite discrete spaces I_i, homeomorphisms φ_i and set function $h\colon I_1 \to I_2$ such that the following diagram commutes:*

$$\begin{tikzcd}
p_1^{-1}(U) \arrow[rrrr, "\phi"] \arrow[dr, "\varphi_1"] \arrow[ddrr, bend right, "q_1"'] & & & & p_2^{-1}(U) \arrow[dl, "\varphi_2"'] \arrow[ddll, bend left, "q_2"] \\
& U \times I_1 \arrow[rr, "(1_U, h)"] \arrow[dr] & & U \times I_2 \arrow[dl] & \\
& & U & &
\end{tikzcd}$$

Proof of Lemma. Note that the intersection of evenly covered open sets is itself open and evenly covered. In particular, there is an non-empty neighbourhood V of x which is evenly covered under both covering maps. This gives all of the diagram besides the existence of h.

We construct h as follows. We define $r = \varphi_2 \circ \phi \circ \varphi_1^{-1}$. It is continuous and since it preserves projections, we must have $r(u, i) = (u, r_u(i))$ for some map $r_u\colon I_1 \to I_2$. We are now looking for a neighbourhood $U \subset V$ of x on which r_u is the same for all $u \in U$. Let us consider the map

$$\begin{aligned} V \times I_1 &\to I_2 \times I_2 \\ (u, i) &\mapsto (r_x(i), r_u(i)) \end{aligned}$$

The diagonal $\Delta = \{(i, i) \in I_2 \times I_2\}$ is open and thus so is its preimage $\{(u, i) \in V \times I_1 \mid r_x(i) = r_u(i)\}$. This set contains $x \times I_1$ and so we may take a neighbourhood $U \times I_1$ within it. Letting $h = r_x$ on this neighbourhood, we are done. $\square$

Now we can show that $Y \times_Z W$ is a covering space. Let $x \in X$. The lemma holds in reference to both f and g and by intersecting open sets we may find a neighbourhood U of x in which the lemma holds for both. Then U is evenly covered by p, since $p^{-1}(U)$ is homeomorphic to $U \times I$ for some discrete, finite I.

Then we just need to know that this is the terminal cone. Since it is the pullback in **Top**, for any other covering space $\mathcal{X}$ and covering space maps ϕ_Y, ϕ_W, there is a unique continuous map $\phi\colon \mathcal{X} \to Y \times_Z W$ such that the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "\phi_Y"] \arrow[ddr, bend right, "\phi_W"'] & & \\
& Y \times_Z W \arrow[r, "\pi_Y"] \arrow[d, "\pi_W"] & Y \arrow[d, "f"] \\
& W \arrow[r, "g"] & Z
\end{tikzcd}$$

commutes. In fact, $\phi = (\phi_Y, \phi_W)$ and so $p \circ \phi(x) = p_Y(\phi_Y(x)) = p_{\mathcal{X}}(x)$ with the last equality following from the fact that ϕ_Y is a covering space morphism. Thus $\phi \in \mathrm{Hom}_{\mathbf{Cov}_X}(\mathcal{X}, Y \times_Z W)$ and is unique. $\square$

Theorem 3.3.3. *Let k be a field. Let A, B, C be separable k-algebras and let $f\colon C \longrightarrow A$ and $g\colon C \longrightarrow B$ be k-algebra homomorphisms. Then the pushout of f and g in ${}_k\mathbf{SAlg}$ exists and is given by the tensor product of A and B over C, $A \otimes_C B$ with the appropriate inclusions $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$.*

Recall. The maps g and f turn A and B into C-modules. It is with this structure that we take the tensor product.

Proof. Take $c \in C$. Then it maps $c \mapsto f(c) \mapsto f(c) \otimes 1_B \in A \otimes_C B$ via A and maps $c \mapsto g(c) \mapsto 1_A \otimes g(c) \in A \otimes_C B$ and with the C-module structure

$$f(c) \otimes 1_B = c \cdot 1_A \otimes 1_B = 1_A \otimes c \cdot 1_B = 1_A \otimes g(c).$$

and so it is a cocone. So we only need that it is the initial cocone. Let us assume we have another cocone, and such the commutative diagram:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\phi_A"] \\
B \arrow[r, "\phi_B"'] & D
\end{tikzcd}$$

Then we may define the map

$$\begin{aligned}\varphi\colon A \times B &\to D \\ (a,b) &\mapsto \phi_A(a)\phi_B(b).\end{aligned}$$

This is bilinear and has the additional property that

$$\varphi(f(c)a, b) = \phi_A(f(c)a\phi_B(b) = \phi_A(f(c))\phi_A(a)\phi_B(b) = \phi_B(g(c)\phi_A(a)\phi_B(b) = \varphi(a, g(c)b)$$

and so $\varphi(c \cdot a, b) = \varphi(a, c \cdot b)$. The universal property of the tensor product then tells us that there is an algebra homomorphism $\phi\colon A \otimes_C B \longrightarrow D$ and that it is the unique such map with $\phi \circ \iota = \varphi$, where $\iota\colon A \times B \to A \otimes_C B$. Thus ϕ is a morphism of cocones and it is unique. □

Corollary 3.3.4. *We have fibre products in ${}_k\mathbf{SAlg}^{\mathrm{op}}$, given by the corresponding pushout in ${}_k\mathbf{SAlg}$.*

3.4 Inverse Limits and Profinite Groups

This last, more abstract example of a limit will be useful later on when we return to field theory.

Recall. A (categorical) poset is a small category $\mathbf{J}$ such that for any $i, j \in \mathrm{Ob}(\mathbf{J})$, there is at most one element in $\mathrm{Hom}_{\mathbf{J}}(i,j) \cup \mathrm{Hom}_{\mathbf{J}}(j,i)$. If $\mathrm{Hom}_{\mathbf{J}}(i,j) \neq \varnothing$, we write that $i \leqslant j$.

Definition 3.4.1. Let $\mathbf{J}$ be a poset and let $A\colon \mathbf{J} \Rightarrow \mathbf{AbGrp}$ be a diagram. In particular, this means that for all $i \leqslant j$, we have a group homomorphism $\mu_{ij}\colon A_i \longrightarrow A_j$ such that if $i \leqslant j \leqslant k$, $\mu_{jk} \circ \mu_{ij} = \mu_{ik}$. Then the limit $\varprojlim A_i$, if it exists, is called *the inverse limit of the system* $\{A_i\}$.

Proposition 3.4.2. *For any such system, $\varprojlim A_i$ exists and is equal (or uniquely isomorphic to) the group*

$$\left\{(a_i)_{i \in J} \in \bigotimes_{i \in J} A_i \mid \mu_{jk}(a_j) = a_k \text{ for all } j \leqslant k\right\}$$

where we notate $J := \mathrm{Ob}(\mathbf{J})$.

Proof. If we have $D \in \mathrm{Ob}(\mathbf{Grp})$ with maps $\pi_i\colon D \longrightarrow A_i$ which makes it into a cone, then we also have the unique morphism of cones

$$\begin{aligned}\pi\colon D &\longrightarrow \varprojlim A_i \\ d &\longmapsto (\pi_i(d))_{i \in I}\end{aligned}$$

□

So again limits give us a way of picking out those elements that satisfy a series of equations.

Definition 3.4.3. Let $\mathbf{J}$ be a poset. We say that $\mathbf{J}$ is *directed* if and only if for all $i, j \in \mathrm{Ob}(\mathbf{J})$, there exists $k \in \mathrm{Ob}(\mathbf{J})$ such that $i \leqslant k$ and $j \leqslant k$.

Definition 3.4.4. Let $\mathbf{J}$ be a directed poset and let $G\colon \mathbf{J} \Rightarrow \mathbf{Grp}$ be a diagram with G_i finite for all $i \in \mathrm{Ob}(\mathbf{J})$. Then the group $G = \varprojlim G_i$ is called *a profinite group.* Giving each G_i the discrete topology, G is a limit in the category of topological groups and inherits a topology.

Chapter 4

Galois Categories

We are now only moments away from define our first truly uniting category theoretical construction. There are only a couple of definitions that remain.

It is helpful for us to understand the properties that functors between categories preserve. This motivates the following definitions:

Definition 4.0.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$. F is called *conservative* if and only if F a morphism $\phi\colon A \longrightarrow B$ is an isomorphism if and only if $F\phi\colon FA \longrightarrow FB$ is an isomorphism.

Remark. If $\phi\colon A \xrightarrow{\sim} B$ is an isomorphism then $\mathbf{1}_{FA} = F(\mathbf{1}_A) = F(\phi^{-1} \circ \phi) = F(\phi^{-1}) \circ F(\phi)$ so $F(\phi)$ is an isomorphism for any functor. The other direction is not generally true.

Since we may compose functors, for a diagram $\mathcal{I}\colon \mathbf{I} \Rightarrow \mathbf{C}$ and a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ determines a diagram with index category $\mathbf{I}$, $F\mathcal{I}\colon \mathbf{I} \Rightarrow \mathbf{D}$. It is meaningful then to ask the relationships between the limits and colimits of $\mathcal{I}$ in $\mathbf{C}$ and $F\mathcal{I}$ in $\mathbf{D}$.

Definition 4.0.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$. F is called *exact* if and only if F commutes with limits and colimits.

When we say that F commutes with limits, we mean that if $\varprojlim D_i$ is the limit of a diagram $D\colon \mathbf{I} \Rightarrow \mathbf{C}$, then $F\left(\varprojlim D_i\right) = \varprojlim FD_i$ where we apply F to each morphism associated with the limit. Similarly for commuting with colimits.

In **Set**, or other concrete categories, we can see injective maps as inclusions. An inclusion $A \hookrightarrow X$ then gives us a way of partitioning $X = X \cup (X \backslash A)$. That motivates our next definition:

Definition 4.0.3. Let $\mathbf{C}$ be a category, and let $f\colon A \longrightarrow B$ be a monomorphism in $\mathbf{C}$. We say f *is an isomorphism of A with direct summand of B* if there exists an element $C \in \mathrm{Ob}\,(\mathbf{C})$ and a morphism $g\colon C \longrightarrow B$ such that B is isomorphic to $A \coprod C$.

Note that these don't exist in all categories. Particularly, in **Top** the coproduct of two non-empty spaces is always disconnected. Then any injective continuous map whose image is not both open and closed is not an isomorphism with direct summand of its codomain.

4.1 The Definition of a Galois Category

Definition 4.1.1 (Galois Category)**.** [12] Let $\mathbf{C}$ be a category and let $F\colon \mathbf{C} \Rightarrow \mathbf{FinSet}$ be a functor. Then we call the pair $(\mathbf{C}, F)$ *a Galois category with fundamental functor* F if it fulfils the following conditions:

1. $\mathbf{C}$ has all finite limits and colimits.
2. F is exact and conservative.
3. For any morphism in $\mathbf{C}$, $f\colon A \longrightarrow C$, there is a $B \in \mathrm{Ob}\,(\mathbf{C})$, an epimorphism $g\colon A \longrightarrow B$ and a monomorphism $h\colon B \longrightarrow C$ such that $f = h \circ g$.
4. Every monomorphism $f\colon A \longrightarrow B$ is an isomorphism of A with direct summand of B.

What initially makes this definition rather intimidating is the same thing that will eventually make it of such use to us. It is both highly abstract and seemingly highly constrained. What is so powerful here is that we find these are the exact properties needed to unite the apparent similarities between the Galois correspondence of covering spaces and the fundamental theorem of Galois field-theory.

Our aim in this chapter is to show that any given Galois category are equivalent to categories of finite G-sets for some profinite group G. Better yet, in our favourite categories of $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}^{\mathrm{op}}$, this equivalence refers exactly to the Galois correspondences from Algebraic Topology and Field theory. In other words, we find that ${}_k\mathbf{SAlg}^{\mathrm{op}} \simeq \mathrm{Gal}(k_s/k)$-$\mathbf{FinSet}$ and if X admits a universal cover $\tilde{X}$, then $\mathbf{Cov}_X \simeq \pi(\tilde{X}, x)$-$\mathbf{FinSet}$.

Before we reach this though, we need to address a practical issue. Conditions 3 and 4 seem fairly simple, but it is not immediately clear how we might go about trying to prove the existence of all limits, or that F is exact.

The trick here is to consider how we might build new limits from old ones. If we can show all limits are examples of a few simple ones, we might be able to reduce out workload. For example, we can build all finite products by iteratively taking pairwise products, so if we show that pairwise products exist in a category, we get all finite products for free. We can note that if our category has a terminal object, the fibre product of any two objects over the terminal object is their product. So if a category has a terminal object and fibre products then it must have all finite products too. It turns out that this is enough:

Theorem 4.1.2. *Let* $\mathbf{C}$ *be a category. Suppose* $\mathbf{C}$ *has a final object* Z *and all fibre products exist in* $\mathbf{C}$*. Then all finite limits exist in* $\mathbf{C}$*.*

We will prove this theorem with the use of the following lemma from [7].

Lemma 4.1.3. *For finite* $\mathbf{I}$ *and a diagram* $D\colon \mathbf{I} \Rightarrow \mathbf{C}$*, if* $\mathbf{C}$ *has all equalisers and further all pairwise products of* D_i*s then* $\varprojlim D_i$ *exists.*

Thus if we can prove that all equalisers and products exist in $\mathbf{C}$, then we know all finite limits exist.

Proof of lemma. Let us denote $I := \mathrm{Ob}\,(\mathbf{I})$. We have the existence of two products:

$$\prod_{i \in I} D_i$$

and

$$\prod_u D_k = \prod_{u\colon j \longrightarrow k} D_k$$

where the lower product is over all the arrows in $\mathbf{I}$ and so has a copy of D_k for every arrow in the diagram which points there. Then they come equipped with projections $p_j\colon \prod_{i\in I} D_i \longrightarrow D_j$ for $i \in I$ and $q_v\colon \prod_u D_k \longrightarrow D_{\mathrm{cod}(v)}$ where $\mathrm{cod}(v)$ is the codomain of v, for arrows in $\mathbf{I}$. Then since $\prod_u D_k$ is a product and for each u we have a map $p_{\mathrm{cod}(u)}\colon \prod_{i\in I} D_i \longrightarrow D_{\mathrm{cod}(u)}$, there exists a unique $f\colon \prod_{i\in I} D_i \longrightarrow \prod_u D_k$ such that $q_u \circ f = p_{\mathrm{cod}(u)}$ for all u.

This is not the only way to give maps $\prod_{i\in I} D_i \longrightarrow D_{\mathrm{cod}(u)}$. Since $Du\colon D_{\mathrm{dom}}(u) \longrightarrow D_{\mathrm{cod}(u)}$, we may take $Du \circ p_{\mathrm{dom}(u)}\colon \prod_{i\in I} D_i \longrightarrow D_{\mathrm{cod}(u)}$ and thus we have a unique $g\colon \prod_{i\in I} D_i \longrightarrow \prod_u D_k$ such that $q_u \circ g = Du \circ p_{\mathrm{dom}(u)}$ for all u.

This is to say, for all arrows u in $\mathbf{I}$, the top and bottom squares of this diagram commute

$$\begin{tikzcd}
D_{\mathrm{cod}(u)} \arrow[r, equal] & D_{\mathrm{cod}(u)} \\
\prod_u D_{\mathrm{cod}(u)} \arrow[u, "q_u"] \arrow[d, "q_u"'] & \prod_{i\in I} D_i \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \arrow[u, "p_{\mathrm{cod}(u)}"'] \arrow[d, "p_{\mathrm{dom}(u)}"] \\
D_{\mathrm{cod}(u)} & D_{\mathrm{dom}(u)} \arrow[l, "Du"]
\end{tikzcd}$$

Now we may form the equaliser

$$\begin{tikzcd}
E \arrow[r, dashed, "e"] & \prod_i D_i \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & \prod_u D_k
\end{tikzcd}$$

Claim. E equipped with the morphisms $\mu_i := p_i \circ e\colon E \longrightarrow D_i$ is a cone of D.

$$\begin{tikzcd} D_i & \\ \prod_{i\in I} D_i \arrow[u, "p_i"] & E \arrow[l, "e"] \arrow[ul, dashed, "\mu_I"'] \end{tikzcd}$$

For any arrow u in $\mathbf{I}$

$$\mu_{\mathrm{cod}(u)} := p_{\mathrm{cod}(u)} \circ e = q_u \circ f \circ e = q_u \circ g \circ e = Du \circ p_{\mathrm{dom}(u)} \circ e =: Du \circ \mu_{\mathrm{dom}(u)}$$

where we use the commutativity of the two squares in the diagram and that e equalises f and g. So the claim is true.

Now let C be another cone, so we have maps $c_i\colon C \longrightarrow D_i$. Then we get our unique map $c\colon C \longrightarrow \prod_{i\in I} D_i$. The fact that this is a cone means that it equalises f and g in which case it factors uniquely through E. Thus E is the limit that we are looking for. $\square$

Proof of theorem 4.1.2. Since we know that $A\times B = A\times_Z B$, where Z is the terminal object, it remains only to show that the existence of fibre products and a final object gives existence of all equalisers. Therefore, let $f, g \in \mathrm{Hom}_{\mathbf{C}}(A, B)$. Construct the fibre product of f and g, $A \times_B A$:

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{4.1}$$

In **Set**, this would be constructing the set $\{(x, y) \in A \times A \mid f(x) = g(y)\}$. Then we construct the fibre product E as:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\varphi"] & A \arrow[d, "(\mathbf{1}_A{,}\mathbf{1}_A)"] \\ A \times_B A \arrow[r, "(\pi_1{,}\pi_2)"'] & A \times A \end{tikzcd} \tag{4.2}$$

Claim. $\phi\colon E \longrightarrow A$ is the equaliser of f and g.

Note that in **Set**, $E = \{((x, y), z) \in (A \times_B A) \times A \mid (x, y) = (z, z)\}$ and so we must have $x = y = z$ and $f(x) = g(y) = g(x)$ for every triple in E and so is the equaliser we are looking for.

In the general setting, we want that $f \circ \phi = g \circ \phi$. Our first diagram gives that $f \circ \pi_2 = g \circ \pi_1$ and we apply the projections from $A \times A$ on the second diagram, alongside the definition of the product of function, to find that $\phi = \pi_1 \circ \varphi = \pi_2 \circ \varphi$. Then

$$f \circ \phi = f \circ \pi_2 \circ \varphi = g \circ \pi_1 \circ \varphi = g \circ \phi$$

so ϕ equalises f and g.

Any $e\colon E' \longrightarrow A$ which equalises f and g gives a cone of eq. (4.1) by putting e on both the top and the left, thus we get a unique morphisms $E' \longrightarrow A \times_B A$. Then this is a cone eq. (4.2) is we let $\phi = e$ and φ be that unique morphism. Thus we have a unique morphism of cones $E' \longrightarrow E$ and E is universal. $\square$

This gives us a number of powerful results:

Corollary 4.1.4. *If a category* $\mathbf{C}$ *has at least one of*

- *all finite coproducts and coequalisers*
- *an initial object and all pushouts*

then it has all finite colimits.

Theorem 4.1.5. *For a functor* $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ *on a category* $\mathbf{C}$ *with all finite limits, the following statements are equivalent:*

1. *F commutes with all finite limits.*

2. *F commutes with finite products and equalisers.*

3. *F maps final objects to final objects and commutes with fibre products.*

Proof. *1.* implies *3.*. Then our proof of theorem 4.1.2 gave us that finite products and equalisers were the result of taking fibre products and using final objects. Thus *3.* implies *2.*. Finally, the proof of lemma 4.1.3 tells us that all limits are the result of taking equalisers and products, so *2.* implies *1.*. □

Corollary 4.1.6. *For a functor $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ on a category $\mathbf{C}$ with all finite colimits, the following statements are equivalent:*

1. *F commutes with all finite colimits.*

2. *F commutes with finite coproducts and coequalisers.*

3. *F maps initial objects to initial objects and commutes with pushouts.*

With these results we are ready to show that we have met some Galois categories already.

4.2 Example: Finite Sets

Theorem 4.2.1. *The category of finite sets,* **FinSet***, is a Galois category with fundamental functor* $\mathbf{1_{FinSet}}\colon \mathbf{FinSet} \Rightarrow \mathbf{FinSet}$.

Proof. We will prove each condition.

1. **FinSet** has both an initial and a final object in $\varnothing$ and $\{\varnothing\}$ (or any singleton) respectively. As we have shown, both fibre products and pushouts exist in **FinSet** and so by theorem 4.1.2 and corollary 4.1.4, **FinSet** has all finite limits and colimits.

2. The identity functor trivially has these properties.

3. In **FinSet**, epimorphisms are surjections and monomorphisms are injections. So let $f\colon A \longrightarrow B$ be a map of sets. The decomposition $A \xrightarrow{f} f(A) \hookrightarrow B$ is as desired.

4. Let $f\colon A \longrightarrow B$ be injective. Then $B = f(A) \coprod (B \backslash f(A))$.

□

4.3 Example: Covering Spaces

Theorem 4.3.1. *Let $X \in \mathrm{Ob}(\mathbf{Top})$ be a connected topological space and let $x \in X$. Let us define $F_x\colon \mathbf{Cov}_X \Rightarrow \mathbf{FinSet}$ to be the functor which takes the covering space $p_Y\colon Y \to X$ to the fibre $p_Y^{-1}(x)$ and takes a morphism $f\colon Y \longrightarrow W$ to the map $f\restriction_{p_Y^{-1}(x)}\colon p_Y^{-1}(x) \to p_W^{-1}(x)$. Then $(\mathbf{Cov}_X, F_x)$ is a Galois category.*

Proof. 1. We have already established in section 1.4.2 that we have both final and initial objects. Further in section 3.3 we showed we have fibre products. All that remains is to show that we have pushouts.

Assume we have morphisms $f_i\colon Z \longrightarrow Y_i$ for $i = 1, 2$. We know that the pushout in **Top** is the set $Y_1 \coprod Y_2/\sim$ where $\sim$ is generated by relations of the form $f_1(z) \sim f_2(z)$ for $z \in Z$, with the associated inclusions $y \mapsto [y]$. Define the map

$$p\colon Y_1 \coprod Y_2/\sim \;\to X$$
$$[y] \mapsto \begin{cases} p_1(y) & \text{if } y \in Y_1 \\ p_2(y) & \text{if } y \in Y_2 \end{cases}$$

Note this is well defined, because if $y_1 \sim y_2$ either $y_1 = y_2$ or there is a $z \in Z$ with $y_i = f_i(z)$ in which case $p_1(f_1(z)) = p_Z(z) = p_2(f_2(z))$ since the f_is are covering space morphisms. In much the same way as with fibre products, including the use of lemma 3.3.2 again, it can be shown that this is a covering space and so the desired pushout.

Thus $\mathbf{Cov}_X$ has all finite limits and colimits.

2. Since $F_x(\mathbf{1}_X) = \mathbf{1}_X^{-1}(x) = x$ and $F_x(\varnothing \to X) = \varnothing$, we have that initial and final objects commute with F. We need then that F commutes with fibre products and pullbacks.

   Let $f_i \colon Y_i \longrightarrow Z$ be morphisms for $i = 1, 2$ and let $Y_1 \times_Z Y_2$ be the fibre product. Then

$$\begin{aligned}
F_x(Y_1 \times_Z Y_2) &= F_x(\{(y_1, y_2) \in Y_1 \times Y_2 \mid f_1(y_1) = f_2(y_2)\}) \\
&= \{(y_1, y_2) \in Y_1 \times Y_2 \mid f_1(y_1) = f_2(y_2), p_{Y_1}(y_1) = p_{Y_2}(y_2) = x\} \\
&= \left\{(y_1, y_2) \in p_{Y_1}^{-1}(x) \times p_{Y_2}^{-1}(x) \mid f_1(y_1) = f_2(y_2)\right\} \\
&= \{(y_1, y_2) \in F_x(Y_1) \times F_x(Y_2) \mid F_x(f_1)(y_1) = F_x(f_2)(y_2)\} \\
&= F_x(Y_1) \times_{F_x(Z)} F_x(Y_2)
\end{aligned}$$

   since $F(f_i)$ is the restriction of f_i to $P_{Y_i}^{-1}(x)$. This is the desired result.

   Let $g_i \colon Z \longrightarrow Y_i$ be morphisms for $i = 1, 2$ and let $W = Y_1 \coprod_Z Y_2 / \sim$ be the pushout.

   Each $[y] \in W$ is in one of $\imath(Y_1) \backslash \imath(Y_2), \imath(Y_2) \backslash \imath(Y_1)$ or $\imath(Y_1) \cap \imath(Y_2)$, so $F_x(W)$ is equal to the union of the elements of the fibre in each of these sets. That is the sets

$$\{[y] \in W \mid y \in Y_i \backslash f_i(Z), p_i(y) = x\}$$

   and the set

$$\{[y] \in W \mid \exists z \in Z, y = f_i(z) \text{ for } i \in \{0, 1\}, p(y) = x\}$$

   form a partition of $F_x(W)$. This is exactly the result of taking $p_1^{-1}(x) \coprod p_2^{-1}(x)$ and gluing it along the image of $p_Z^{-1}(x)$, since $p_Z(z) = x$ if and only if $p_i(f_i(z)) = x$.

   Thus $F_x(W)$ is exactly the pushout of the diagram

$$\begin{tikzcd}
F_x(Z) \arrow[r, "F(f_1)"] \arrow[d, "F(f_2)"'] & F_x(Y_1) \\
F_x(Y_2) &
\end{tikzcd}$$

   and we are done.

   So we have that F is exact.

   The proof that F is conservative is from [10].

   Let $\phi \colon Y_1 \longrightarrow Y_2$ be a morphism of covering spaces such that $F_x\phi \colon p_1^{-1}(x) \longrightarrow p_2^{-1}(x)$ is a bijection. This means that the set function h constructed in lemma 3.3.2 is a bijection on some neighbourhood of x and consequently the set of $y \in X$ for which $F_y\phi \colon p_1^{-1}(y) \longrightarrow p_2^{-1}(y)$ is invertible is also open, and contains x. A similar argument shows its complement is also open, but X is connected so the complement is empty. Thus ϕ is a continuous bijection.

   Lastly, evenly covered opens form a basis of the topology on X and ϕ respects the covering maps, we must have that ϕ is open and thus a homeomorphism.

3. Let $f \colon Y_1 \longrightarrow Y_2$ be a morphism of covering spaces.

   Let us define the space $Y = f(Y_1) \subset Y_2$. This is a covering space, inheriting $p_2\restriction_Y$.

   Then we may define

$$\begin{aligned}
f'' \colon Y_1 &\longrightarrow Y \\
y_1 &\longmapsto f(y_1)
\end{aligned}$$

   and

$$\begin{aligned}
f' \colon Y &\longrightarrow Y_2 \\
y_2 &\longmapsto y_2
\end{aligned}$$

   Then, as epimorphism and monomorphisms are just surjections and injections from section 1.4.1, f'' is epic, f' is monic and $f = f' \circ f''$.

4. Let $f: Y_1 \longrightarrow Y_2$ be injective. If the image of f is both open and closed then it is a connected component of Y_2, then $Y_2 \backslash f(Y_1)$ is still a covering space and we would have $Y_2 = f(Y_1) \coprod (Y_2 \backslash f(Y_1))$. So let us show that $f(Y_1)$ is open.

   Take $y_2 \in f(Y_1)$. Then there is exactly one $y_1 \in Y_1$ with $f(y_1) = y_2$. There exists a neighbourhood of $x := p_1(y_1) = p_2(y_2)$ which is evenly covered by both covering spaces. Then the copy of this neighbourhood around y_1 is mapped homeomorphically onto the copy of this neighbourhood around y_2, again using lemma 3.3.2 and thus this open is contained in $f(Y_1)$ and it is open.

   Let us take $y_2 \in Y_2 \backslash f(Y_1)$ such that every neighbourhood of y_2 intersects $f(Y_1)$. Let us take a neighbourhood U of $x := p_{Y_2}(y_2)$ which is evenly covered by both covering spaces such that lemma 3.3.2 holds. Then one of the corresponding neighbourhoods in Y_1 intersects the one around y_2 and so must map onto this neighbourhood and $y_2 \in f(Y_1)$. So $f(Y_1)$ contains its boundary, is both open and closed and we are done.

□

4.4 Example: Separable Algebras

Now we return to our category ${}_k\mathbf{SAlg}^{\mathrm{op}}$. For this section, we need some results from Galois field theory, which can be found in appendix A.2, and a brief review of the important definition of group actions, found in appendix A.3. As in section 1.5, we will use these as tools to better understand the categorical apparatus we have set up.

Now, nearly 40 pages on, we return to our motivating theorem theorem 0.0.1:

Theorem 0.0.1. *Let k be a field. Then the category ${}_k\mathbf{SAlg}$ of separable k-algebras is anti-equivalent to the category $\mathrm{Gal}(k_s/k)$-$\mathbf{FinSet}$ of finite sets with a continuous action of $\mathrm{Gal}(k_s/k)$, where k_s is the separable closure of k.*

Proof. [6] If B is a free separable k-algebra, let $F(B) = {}_k\mathbf{Alg}(B, k_s)$, the set of all k-algebra homomorphisms $B \to k_s$. This is finite because B is separable.

If $g \in F(B)$ and $\sigma \in \mathrm{Gal}(k_s/k)$, then $\sigma \circ g: B \longrightarrow k_s$ is a k-algebra homomorphism so we get an action of $\mathrm{Gal}(k_s/k)$ on $F(B)$. This action is continuous.

If $f: B \longrightarrow C \in {}_k\mathbf{SAlg}(B, C)$, then define $F(f): F(C) \longrightarrow F(B)$ by $F(f)(g) = g \circ f$, where $g: C \longrightarrow k_s$.

This is a contravariant functor. It fulfils the conditions we required for the first definition of an equivalence. A full description of a weak inverse of F can be found in [6]. □

There is something unsatisfying about this field theory heavy proof of our statement, especially given how closely it resembles theorem 0.0.2 in statement. This is because they are both examples of a larger theorem about Galois categories. Thus to complete our examples, we have:

Theorem 4.4.1. *For a field, k, let ${}_k\mathbf{SAlg}$ be the category of separable k-algebras. Then $({}_k\mathbf{SAlg}^{\mathrm{op}}, F)$ for $F = {}_k\mathbf{Alg}(-, \bar{k})$ is a Galois category.*

We will state without proof two lemmas from [10] which will be helpful here:

Lemma 4.4.2. *Let $A = \oplus_{i=1}^m A_i$ decomposed into finite separable field extensions as in theorem A.1.5. Then for any field L, any k-algebra homomorphism $A \longrightarrow L$ factors through some projection $\pi_i: A \longrightarrow A_i$.*

Lemma 4.4.3. *Let $A = \oplus_{i=1}^m A_i$ and $B = \oplus_{j=1}^n B_j \neq k^0$ be separable k-algebras, decomposed into finite separable field extensions as in theorem A.1.5. Then as sets,*

$$ {}_k\mathbf{Alg}(A, B) = \mathrm{Hom}_{{}_k\mathbf{Alg}}(A, B) \cong \prod_{j=1}^{n} \coprod_{i=1}^{m} \mathrm{Emb}_k(A_i, B_j). $$

This has the important corollary that if ${}_k\mathbf{Alg}(A, B)$ is non-empty, then for every j there is some i such that A_i embeds into B_j.

Proof of Theorem. 1. We have from section 1.5 and section 3.3 that initial and final objects exist and so do fibre products. We need only the existence of pushouts, which is to say the existence of fibre products in ${}_k\mathbf{SAlg}$.

   Let us take $f: A \longrightarrow C$ and $g: B \longrightarrow C$. Then the pullback does indeed exist and is given by $A \times_C B = \{(a, b) \in A \oplus B \mid f(a) = g(b)\}$ as we have come to expect. The fact that this is separable is slightly non-trivial and given in [10].

2. $F(k) = {}_k\mathbf{Alg}(k, \bar{k}) = \{1\}$ so we have that F takes final objects to final objects. Since $F(k^0) = \varnothing$, it sends initial objects to initial objects.

   So we need that F sends fibre products to pushouts and visa versa. For the pushout of $f\colon C \longrightarrow A$ and $g\colon C \longrightarrow B$:

$$\begin{aligned} F(A \otimes_C B) &= {}_k\mathbf{Alg}(A \otimes_C B, \bar{k}) \\ &= \big\{C\text{-bilinear maps } \Phi\colon A \times B \to \bar{k}\big\} \\ &= \big\{\Phi\colon A \times B \to \bar{k} \mid \Phi(c \cdot a, b) = c \cdot \Phi(a, b) = \Phi(a, c \cdot b)\big\} \\ &= \big\{(\phi, \varphi) \in {}_k\mathbf{Alg}(A, \bar{k}) \times {}_k\mathbf{Alg}(B, \bar{k}) \mid F(f)(\phi) = F(g)(\varphi)\big\} \\ &= F(A) \times_{F(C)} F(B). \end{aligned}$$

   A similar statement is shown in [10] for sending fibre products to pushouts. We will take it for granted so as to note that F is exact.

   Finally, if $F(u)\colon {}_k\mathbf{Alg}(A, \bar{k}) \longrightarrow {}_k\mathbf{Alg}(B, \bar{k})$ is an isomorphism, then we must have that u is an isomorphism, so F is conservative.

3. Let $\phi\colon A \longrightarrow B$ be a morphism in ${}_k\mathbf{SAlg}$. Write $A = \oplus_{i=1}^m$ and $B = \oplus_{j=1}^n$. We can write $\phi = (\phi_1, \ldots, \phi_n)$, for $\phi_j = \pi_j \circ \phi$. By lemma 4.4.2, we have that each ϕ_j factors through some $A_{i(j)}$. Then since $\phi_j\left(A_{i(j)}\right)$ is a subfield of B_j, $\oplus_{j=1}^n \phi_j(A_{i(j)})$ is a separable k-algebra. Thus we have the decomposition into epic and monic morphisms:

$$\bigoplus_{i=1}^m A_i \longrightarrow \bigoplus_{j=1}^n \phi_j(A_{i(j)}) \hookrightarrow \bigoplus_{j=1}^n B_j.$$

   Since when we dualise this category we will reverse all arrows and swap monomorphisms and epimorphisms, this is still an adequate decomposition.

4. A monomorphism in ${}_k\mathbf{SAlg}^{\mathrm{op}}$ is an epimorphism in ${}_k\mathbf{SAlg}$. This is just a surjective morphism. So we begin with a surjective morphism $f\colon A \longrightarrow B$ and we hope to find another surjective morphism $g\colon A \longrightarrow C$ such that $A \cong B \oplus C$, where $\oplus$ is the product in ${}_k\mathbf{SAlg}$ from *section* 1.5. Then let us decompose A and B as above. From lemma 4.4.3, we have that there are isomorphisms of fields $A_{i(j)} \longrightarrow B_j$ for all j and thus $B \cong \oplus_{i=1}^k A_i$, for some reordering of the A_is and $k \leqslant m$. Then we may take $C = \oplus_{i=k+1}^m A_i$ with the surjective projection $A \longrightarrow C$ and note $A = B \oplus C$. □

4.5 Properties

So we have established a definition of Galois categories, and shown that some of our favourite categories are examples of them. What now? We could be forgiven for getting a little bored in the last few pages. Our definition was general enough that it hasn't shed any immediate light on the similarities between these categories but specific enough that we've had to put in some work to verify that $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}^{\mathrm{op}}$ are indeed Galois categories. Thankfully though, with that legwork behind us we are ready for the payoff.

Galois categories have a number of interesting, small properties. For example

Proposition 4.5.1. *Let* $(\mathbf{C}, F)$ *be a Galois category. Then* $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ *is a monomorphism or an epimorphism if and only if* Ff *is injective or surjective respectively.*

Our concern, however, is to finally understand the connection between theorem 0.0.1 and theorem 0.0.2 and since it is in sight now, we will move on from these small results onto bigger things.

Definition 4.5.2. Let $F\colon \mathbf{C} \Rightarrow \mathbf{D}$ be a functor. *The automorphism group of* F, $\mathrm{Aut}F$ is the set of natural isomorphisms $\alpha\colon F \Rightarrow F$.

Note that in a general case, we may concern ourselves with whether this is a set, but in our examples it is not a concern. In fact, where it is a set, we have a much stronger result:

Theorem 4.5.3. *Let* $F\colon \mathbf{C} \Rightarrow \mathbf{FinSet}$ *be a functor. Then* $\mathrm{Aut}F$ *is canonically a profinite group.*

Proof. Any automorphism $\alpha\colon F \Rightarrow F$ consists of isomorphisms $\alpha_X\colon F(X) \xrightarrow{\sim} F(X)$ for all $X \in \mathrm{Ob}(\mathbf{C})$. In **FinSet**, isomorphisms are just permutations, so α consists of choice of permutation of $\alpha_X \in S(F(X))$ for all $X \in \mathrm{Ob}(\mathbf{C})$ subject to the naturality condition eq. (2.3):

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & F(X) \arrow[d, "F(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & F(Y) \end{tikzcd}$$

That is to say

$$\mathrm{Aut}F \subset \prod_{X \in \mathrm{Ob}(\mathbf{C})} S(F(X))$$

where the right hand side is a profinite group when given the discrete topology in each component. Finally, we note that $\mathrm{Aut}F$ is the intersection of all sets $\{(\alpha_X)_{X\in\mathrm{Ob}(\mathbf{C})} \mid F(f) \circ \alpha_Y = \alpha_Z \circ F(f)\}$ over all morphisms $f\colon Y \longrightarrow Z$ in $\mathbf{C}$. These are closed and thus, as the closed subgroup of a profinite group, $\mathrm{Aut}F$ is profinite . $\square$

Remark. For any $X \in \mathrm{Ob}(\mathbf{C})$, we may form the projection $\mathrm{Aut}F \to S(F(X))$ so in particular we have a continuous action

$$\begin{aligned} \mathrm{Aut}F \times F(X) &\to F(X) \\ (\alpha, x) &\mapsto \alpha_X(x). \end{aligned}$$

Better yet, if $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$, then by naturality $F(f) \circ \alpha_X = \alpha_Y \circ F(f)$ which is to say that for all $x \in X$ and $\alpha \in \mathrm{Aut}F$,

$$F(f)(\alpha_X(x)) = \alpha_Y(F(f)(x))$$

so $F(f)$ is a morphism of $\mathrm{Aut}F$-sets.

So, we may consider the extension of F to $\mathcal{F}\colon \mathbf{C} \Rightarrow \mathrm{Aut}F$-**FinSet**.

A little fiddling around will show us that in our example of $\mathbf{Cov}_X$, if X has a universal cover then $\mathrm{Aut}F_x \cong \pi(X, x)$, the fundamental group of X, and if we consider ${}_k\mathbf{SAlg}^{\mathrm{op}}$, then $\mathrm{Aut}F \cong \mathrm{Gal}(k_s/k)$. So we can see that we have arrived at our grand unifying theorem, the Galois correspondence.

Theorem 4.5.4 (The Galois Correspondence)**.** *Let* $(\mathbf{C}, F)$ *be a Galois category. Then the functor* $\mathcal{F}\colon \mathbf{C} \Rightarrow \mathrm{Aut}F$-**FinSet** *is an equivalence.*

Before we take on this proof, we need a couple more definitions and some lemmas:

Definition 4.5.5. Let $\mathbf{C}$ be any category and $X \in \mathrm{Ob}(\mathbf{C})$. *A subobject of* X is an equivalence class of monomorphisms $m\colon M \longrightarrow X$ under the equivalence relation $m \sim m'$ if and only if there are some f, f' such that

$$\begin{tikzcd} M \arrow[rr, shift left, "f"] \arrow[dr, "m"'] & & M' \arrow[ll, shift left, "f'"] \arrow[dl, "m'"] \\ & X & \end{tikzcd}$$

commutes. For a subobject $[m\colon M \longrightarrow X]$ of X, *a complement of* $[m]$ is second subobject $[m'\colon M' \longrightarrow X]$ such that $X = M \coprod M'$. This behaves nicely with the equivalences.

We may be able to consider this in a clearer way: The category of monomorphisms with codomain X forms a full subcategory of the slice category $\mathbf{C}/X$. Then if we have f and f' above, note that in particular this means

$$m \circ \mathbf{1}_M = m \circ f' \circ f.$$

Since m is a monomorphism, this means

$$f' \circ f = \mathbf{1}_M.$$

The same argument going the other way shows that $f \circ f' = \mathbf{1}_{M'}$, so $f' = f^{-1}$. So the subobjects of X are just the isomorphism classes of monomorphisms in $\mathbf{C}/X$. We are interested in the isomorphism classes because we care only about how the monomorphism fits into X. For example, there is an injective map $\{n\} \to \{a, b\}$ for every $n \in \mathbb{N}$, but there are only 2 singleton subsets they can be mapping to. We don't care about n. We take equivalence classes to match this notion.

Remark. Condition 4. of definition 4.1.1 of a Galois category is equivalently stated as

4'. Every subobject in $\mathbf{C}$ admits a complement.

Definition 4.5.6. An object $X \in \mathrm{Ob}(\mathbf{C})$ is called *connected* if it has exactly two subobjects.

Note that any object X has itself and the initial object as subobjects and they are distinct if and only if X is not initial.

Lemma 4.5.7. *Every non-initial object X in a Galois category $\mathbf{C}$ can be written, unique up to re-ordering, as a finite coproduct of connected objects.*

Proof of lemma 4.5.7. [12] If X admits a subobject $Y \longrightarrow X$ which is neither X nor initial, then it has a complement $Z \longrightarrow X$. Further, because F is exact, this means $F(X) = F(Y) \coprod F(Z)$ where all these sets are non-empty. Since $F(X)$ is finite, we can induct and know that this process will terminate, so X is the coproduct of finitely many connected components. □

Definition 4.5.8. Let $C \in \mathrm{Ob}(\mathbf{C})$ be an object in a locally small category. Then the group of isomorphisms in $\mathrm{Hom}_{\mathbf{C}}(C, C)$ is called *the group of automorphisms of C*, denoted $\mathrm{Aut}C$.

If $G \subset \mathrm{Aut}C$ is a subgroup, then we may define *the quotient of C by G*, denoted C/G, if it exists, as the coequaliser of all $\sigma \in G$.

Lemma 4.5.9. *Let $(\mathbf{C}, F)$ be a Galois category. Then for every connected $C \in \mathrm{Ob}(\mathbf{C})$,*

$$|\mathrm{Hom}_{\mathbf{C}}(C, X)| \leqslant |F(X)|.$$

Proof. Let $C \in \mathrm{Ob}(\mathbf{C})$ be connected and $c \in F(C)$. Then for any $X \in \mathrm{Ob}(\mathbf{C})$, define the map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(C, X) &\to F(X) \\ f &\mapsto F(f)(c). \end{aligned} \tag{4.3}$$

This map is inject: if $F(f)(c) = F(g)(c)$, then let B be the equaliser of f and g. By proposition 3.2.4, $B \longrightarrow C$ is a subobject, and has $c \in F(B)$. In fact, it is the largest subobject which equalises f and g. By connectedness $B = C$, $e = \mathbf{1}_C$ and so $f = g$. □

Then for any connected object $C \in \mathrm{Ob}(\mathbf{C})$, $\mathrm{Aut}C \subset \mathrm{Hom}_{\mathbf{C}}(C, C)$, so is finite. Since $\mathbf{C}$ has all finite colimits, then $C/\mathrm{Aut}C$ exists for all connected C.

Definition 4.5.10. Let us assume that $(\mathbf{C}, F)$ is a Galois category. An object C is said to be *Galois* if it is connected and $C/\mathrm{Aut}C$ is the final object.

Lemma 4.5.11. *An object $C \in \mathrm{Ob}(\mathbf{C})$ is Galois if and only if $\mathrm{Aut}C$ acts transitively on $F(C)$.*

Proof. Since F commutes with colimits, we have that $F(C/\mathrm{Aut}C) = F(C)/\mathrm{Aut}C$ and this is a singleton (final) if and only if $\mathrm{Aut}C$ acts transitively on $F(C)$. □

Lemma 4.5.12. *If an object $C \in \mathrm{Ob}(\mathbf{C})$ is Galois, then the action of $\mathrm{Aut}C$ on $F(C)$ is free.*

Proof. Since for a connected object C, by lemma 4.5.9, we have that $|\mathrm{Aut}C| \leqslant |F(X)|$, if C is Galois we must have $|\mathrm{Aut}C| = |F(X)|$ by lemma 4.5.11. Consequently, the action of $\mathrm{Aut}C$ on $F(C)$ is also free. □

Proof of theorem 4.5.4. From [12]: Take $X \in \mathrm{Ob}(\mathbf{C})$. We define $D := \prod_{x \in F(X)} X$ and $c := \mathrm{id}_{F(X)} \in F(D) = \prod_{x \in F(X)} F(X)$. That is, c has x in the xth place.

Let $C \longrightarrow D$ be the largest connected subobject with $c \in F(C)$. For $x \in F(X)$, let $f_x \colon C \longrightarrow X$ be the projection on the xth factor (where we compose $C \longrightarrow D$ with $\pi_x \colon D \longrightarrow X$. Then $F(f_x)(c) = x$ so eq. (4.3) is bijective for these C, c and X.

Claim. C is Galois.

Proof of claim. Let $c' \in F(C)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(C, X) &\to F(X) \\ f &\mapsto F(f)(c') \end{aligned}$$

is an injection of finite sets with equal cardinality, and so is surjective. Since, by cardinality, $\mathrm{Hom}_{\mathbf{C}}(C, X) = \{f_x \mid x \in F(X)\}$, this means that $c' \in F(D)$ is a permutation of $F(D)$. Then there is an induced automorphism of D, σ, permuting the copies of x such that $c \mapsto c'$ and so C maps to a connected component $C' \longrightarrow D$. Since $c' \in F(C) \cap F(C')$, we find that $C = C'$ and consequently σ restricts to an automorphism of C. //

Thus, for any $X \in \mathrm{Ob}(\mathbf{C})$, there is a Galois object C and element $c \in F(C)$ such that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(C,X) &\to F(X) \\ f &\mapsto F(f)(c') \end{aligned}$$

is a bijection.

Claim. If X is connected, then the right action of $\mathrm{Aut}C$ on $\mathrm{Hom}_{\mathbf{C}}(C,X)$ is transitive.

Proof of claim. Take $f, g\colon C \longrightarrow X$. Because f can be decomposed into the composition of an epic and a monic morphism, and X is connected, we have that f is epic and so $F(f)$ is surjective. So we can choose $d \in F(C)$ with $F(f)(d) = F(g)(c)$, and we choose $\sigma \in \mathrm{Aut}C$ such that $F(\sigma)(c) = d$, by bijectivity. Then because F is functorial $F(f \circ \sigma)(c) = F(g)(c)$ and so $f \circ \sigma = g$. //

Let us continue with connected X. Fix $f\colon C \longrightarrow X$ and define $G := \{\sigma \in \mathrm{Aut}C \mid f \circ \sigma = f\}$.

Claim. The induced map $\hat{f}\colon C/G \longrightarrow X$ is an isomorphism.

Proof of claim. Since F reflects isomorphisms, it is sufficient to show that $F(\hat{f})\colon F(C)/G \to F(X)$ is bijective.

$F(f)$ is surjective, so this map is too. Then since G acts freely on $F(C)$, we have

$$|F(C)/G| = |F(C)|/|G| = [\mathrm{Aut}C\colon G].$$

but $\mathrm{Aut}C$ acts transitively on $\mathrm{Hom}_{\mathbf{C}}(C,X)$ with stabiliser G, so

$$|F(X)| = |\mathrm{Hom}_{\mathbf{C}}(C,X)| = [\mathrm{Aut}C : G]$$

so $F(\hat{f})$ is surjective on sets of the same cardinality. Hence it is bijective. //

The culmination of this part of the proof, then, is the conclusion that every connected object X is the quotient of a Galois object C by a finite subgroup of $\mathrm{Aut}C$.

Now we introduce a new category. Let $\mathbf{I}$ be the category of pairs (C,c), for $C \in \mathrm{Ob}(\mathbf{C})$ Galois and $c \in F(C)$. We define morphisms $(C,c) \longrightarrow (D,d)$ as $f \in \mathrm{Hom}_{\mathbf{C}}(C,D)$ such that $F(f)(c) = d$. Note that again by lemma 4.5.9, any such morphism exists between any two pairs, then it is unique. So $\mathbf{I}$ is a poset. Further, for any $(C,c),(D,d) \in \mathrm{Ob}(\mathbf{I})$, there is a Galois object E, map $f\colon E \longrightarrow C \times D$ and $e \in F(E)$ such that $F(f)(e) = (c,d)$. By projecting, we see that (E,e) is a common bound of (C,c) and (D,d) so in fact $\mathbf{I}$ is a directed set.

Using this, it can be shown (see [12]) that

$$F = \mathrm{colim}_{(C,c)}\mathrm{Hom}_{\mathbf{C}}(C,-)$$

where this colimit varies over all $(C,c) \in \mathrm{Ob}(\mathbf{I}^{\mathrm{op}})$.

Let $f \in \mathrm{Hom}_{\mathbf{I}}((C,c),(D,d))$. $\mathrm{Aut}D$ acts freely and transitively on $F(D)$ and so for each $\sigma \in \mathrm{Aut}C$ there is a unique $\tau \in \mathrm{Aut}D$ such that $F(\tau)(d) = F(f \circ \sigma)(c)$ and thus $\tau \circ f = f \circ \sigma$ (since $F(f)(c) = d$). This gives us a homomorphism $\mathrm{Aut}C \to \mathrm{Aut}D$ which is surjective since $\mathrm{Aut}C$ acts transitively on $\mathrm{Hom}_{\mathbf{C}}(C,X)$.

Now let $\alpha \in \mathrm{Aut}F$. There is a unique $\sigma \in \mathrm{Aut}C$ with $F(\sigma)(c) = \alpha_C(c)$, and so we define a group homomorphism $\mathrm{Aut}F \to \mathrm{Aut}C$. Using these, we may set up a map $\mathrm{Aut}F \to \lim_{(C,c)\in\mathrm{Ob}(\mathbf{I})}\mathrm{Aut}C$ since for $f\colon (C,c) \longrightarrow (D,d)$

$$\begin{tikzcd} & \mathrm{Aut}C \arrow[d, "f_*"] \\ \mathrm{Aut}F \arrow[ur] \arrow[dr] & \\ & \mathrm{Aut}D \end{tikzcd}$$

commutes.

Since all objects of $\mathbf{C}$ are coproducts of connected objects and all connected objects of $\mathbf{C}$ are quotients of Galois objects, any $\alpha \in \mathrm{Aut}F$ is determined by its action of $F(C)$ for C Galois. Hence $\mathrm{Aut}F$ is a closed subgroup of $\prod_{(C,c)\in\mathrm{Ob}(\mathbf{I})} S(F(C))$. Finally, the compatibility condition to be in $\mathrm{Aut}F$ is exactly that required to be in $\lim_{(C,c)\in\mathrm{Ob}(\mathbf{I})}\mathrm{Aut}C$ and so

$$\mathrm{Aut}F = \lim_{(C,c)\in\mathrm{Ob}(\mathbf{I})}\mathrm{Aut}C$$

as profinite groups. The projections $\mathrm{Aut}F \to \mathrm{Aut}C$ are surjective.

F and the forgetful functor $\mathrm{Aut}F$-**FinSet** $\Rightarrow$ **FinSet** are both exact and conservative, and therefore so is $\mathcal{F}$.

Claim. If $X \in \mathrm{Ob}\,(\mathbf{C})$ is connected, then so is $\mathcal{F}(X)$.

Proof of claim. Let X be connected. then $X = C/G$ for some Galois C and G some finite subgroup of $\mathrm{Aut}C$. Then

$$\mathcal{F}(X) = \mathcal{F}(C)/G = \mathrm{Aut}C/G.$$

This is a transitive $\mathrm{Aut}F$-set, so $\mathcal{F}(X)$ is connected. //

So finally we get to proving the equivalence. We use the choice-reliant definition:

Claim. $\mathcal{F}$ is essentially surjective.

Proof of claim. Let P be a finite $\mathrm{Aut}F$-set. Without loss of generality, we may assume that P is transitive. Then $P \cong \mathrm{Aut}C/G$ for some Galois object C and G a finite subgroup of $\mathrm{Aut}C$. Then $P \cong \mathcal{F}(C/G)$ as above. //

Claim. $\mathcal{F}$ is fully faithful.

Proof of claim. This is exactly the statement that for all $X, Y \in \mathrm{Ob}\,(\mathbf{C})$, the map from $\mathcal{F}$, $\mathrm{Hom}_{\mathbf{C}}(X,Y) \to \mathrm{Hom}_{\mathbf{FinSet}}(\mathcal{F}(X), \mathcal{F}(Y))$ is bijective.

It can be shown that these sets have the same cardinality, by arguing on connected components and considering cosets. This can be found in [12].

Since $\mathcal{F}$ commutes with equalisers, morphisms that are not equal on the left will not be equal on the right, so we have injectivity. Thus the map is bijective. //

We have fulfilled all the required conditions and $\mathcal{F}$ is an equivalence. □

It is encouraged to go through this proof step by step with the covering space example in mind. The idea of connected object conforms with that for connected spaces, as does Galois objects and the standard definition for Galois covering spaces. What is so remarkable here is the ability to keep the essence of the necessary properties whilst removing the geometric context, and so widening the scope of the result dramatically, even when keeping the method of the proof almost exactly the same.

4.6 Concluding Galois Categories

With the completion of this proof, we finally see the reward of all the work we have to done to get to this point. In quite a fantastic way, we have used a heavy dose of abstraction to take two already rather involved mathematical examples and unite them. Not only did we do this by finding some class that they both fit into but in fact we found the exact conditions required to express their common behaviours. So as we hoped from the beginning, we have established a language that allows us connections we might have intuitively noticed but had not been able to formalise.

Let us prove one of those first results to provide a satisfying conclusion to this chapter:

Theorem 0.0.2. *Let X be a topological space. Then there is a canonical profinite group $\hat{\pi}(X,x)$ for any $x \in X$ such that the category $\mathbf{Cov}_X$ of finite covering spaces of X is equivalent to the category $\hat{\pi}(X,x)$-$\mathbf{FinSet}$ of finite sets with continuous action of $\hat{\pi}(X,x)$.*

Further, if X admits a universal cover, then $\hat{\pi}(X,x)$ is exactly the profinite completion of $\pi(X,x)$, the fundamental group of X. If the cover is finite, then this is exactly $\pi(X,x)$.

Proof. We will prove the case when X admits a finite universal cover $\hat{p}\colon \hat{X} \to X$. This means that $\hat{X}$ is simply connected and for every covering space $p_Y : Y \to X$, there is a morphism $q_Y : \hat{X} \to Y$ which is itself a covering space of Y.

We know $\mathbf{Cov}_X \simeq \mathrm{Aut}F_x$-$\mathbf{FinSet}$, where $x \in X$, so let us understand what automorphisms of F_x look like. An automorphism $\alpha\colon F \Rightarrow F$ consists of an isomorphism for each covering space $p_Y : Y \to X$ from $p_Y^{-1}(x)$ to itself. Fix α. Then we have a permutation $\alpha_{\hat{X}} : p_{\hat{X}}^{-1}(x) \to p_{\hat{X}}^{-1}(x)$. Let $p_Y : Y \to Y$ be any other coverings space and let $q_Y : \hat{X} \to Y$ be a surjective covering space. Then naturality of α gives us the commutative diagram

$$\begin{tikzcd} p_{\hat{X}}^{-1}(x) \arrow[r, "\alpha_{\hat{X}}", "\sim"'] \arrow[d, "q_Y"] & p_{\hat{X}}^{-1}(x) \arrow[d, "q_Y"] \\ p_Y^{-1}(x) \arrow[r, "\alpha_Y", "\sim"'] & p_Y^{-1}(x). \end{tikzcd}$$

Since q_Y is still surjective when restricted to fibres, it has a right inverse, and so we find $\alpha_Y = q_Y \circ \alpha_{\hat{X}} \circ q_Y^{-1}$. Thus we have that $\alpha\colon F \Rightarrow F$ is fully determined by $\alpha_{\hat{X}}$. Thus we have that $\mathrm{Aut}F$ is equal to the set of possible permutations of $p_{\hat{X}}^{-1}(x)$. Morphisms between connected covering spaces are determined entirely by their action on any single fibre [1] so this is the same as the group of isomorphisms $\phi\colon \hat{X} \xrightarrow{\sim} \hat{X}$: the group of deck transformations of $\hat{X}$. It is a well known result (see [4]) that this is isomorphic to the fundamental group $\pi(X, x)$ of X at x and we are done. $\square$

It may seem now that this is conclusion for Galois categories, a sort of fundamental theorem that solves the branch of maths, but in fact it only opens more options. In particular, as explored in [12], it becomes useful in the study of schemes. A full understanding of what we are doing when we develop the theory of covering spaces and the Galois correspondence associated with it means that in the context of more general geometric objects where we may be unable to define a fundamental group in terms of loops, we are instead granted the alternative of approaching it with Galois categories. In fact, this is exactly the motivation Grothendieck had in developing the original theory. He was successful in his attempt to transfer this powerful tool from algebraic topology to algebraic geometry with the development of the Étale fundamental group.

The perspective developed here did not only does yield its own set of powerful algebraic tools, but also opened the way for further abstractions and developments. It is one of these that we will explore for the rest of this paper.

[1] This is a consequence of Proposition 1.34 in [4]

Chapter 5

Affine Group Schemes

The remainder of this paper will be focused on establishing enough background to understand the definition of another equivalence. Again we will be looking to establish a category of geometric/algebraic objects and showing to be equivalent to a categorical one. In this way, we again find the minimal categorical properties that define our algebraic category. Inspired by chapter 4, again we will have a category with a fundamental functor of sorts which induces the equivalence, but the type of categories and the form of the functor with be radically different.

After the rather technical proofs of the last chapter, we will be toning that down, instead focusing on establishing definitions and key results. Our aim in the rest of the paper is not to prove to the reader the truth of our discussion (this is done very well and in great detail in [3] and [11]), but instead to develop enough of the definitions of both affine group schemes and neutral Tannakian categories such that when we state our final theorem, we understand enough to be surprised and impressed by the result. So with this understood, let's begin.

5.1 The Definition of Affine Group Schemes

It is not unusual for us to construct groups from other algebraic objects. Given a ring R, we can look at the underlying additive group, or consider the multiplicative group of units. We might construct groups of matrices, or the multiplicative group of roots of unity. In all these cases, we are choosing subsets of elements of R which satisfy certain sets of polynomial equations. If we want the multiplicative group of R, then we look for pairs of elements $(x_1, x_2) \in R^2$ which satisfy $x_1 x_2 = 1_R$. For the multiplicative group of 2x2 matrices with unit determinant we want quadruples $(x_1, x_2, x_3, x_4) \in R^4$ such that $x_1 x_4 - x_2 x_3 = 1_R$, where we then attach a group operation to the set of such quadruples.

Let us try to formalise this. All the equations in the above discussion had coefficients in $\{0, 1\}$. Let us do one better: Let k be a field and let R be a k-algebra. This allows us to makes sense of evaluating polynomials with coefficients in k at elements of R.

Let $P \subset k[X_1, \ldots, X_n]$ be a set of polynomials and let $F(R)$ denote the set of n-tuples of elements of R which are solutions to the elements of P. Consider an algebra homomorphism, $\phi\colon k[X_1, \ldots, X_n] \to R$. Naturally such a map picks an n-tuple of elements of R, $(\phi(X_i))_{i\in\{1,\ldots,n\}}$. What does it mean for this n-tuple to be a solution to the equation in P?

For any $f \in P$, we must have $\phi(f) = 0$, so the ideal generated by P is a subideal of the kernel of ϕ. So ϕ must factor through the quotient $k[X_1, \ldots, X_n]/\langle P\rangle$.

Claim. Given a field k, a k-algebra R, and a set of polynomials $P \subset k[X_1, \ldots, X_n]$, the solutions to P in R, denotes $F(R)$, has a natural bijective correspondence with k-algebra homomorphisms ${}_k\mathbf{Alg}(A, R)$ where $A = k[X_1, \ldots, X_n]/\langle P\rangle$.

Proof. We define

$$\begin{aligned} {}_k\mathbf{Alg}(A, R) &\to F(R) \\ \phi &\mapsto (\phi(X_i))_{i\in I} \end{aligned} \tag{5.1}$$

The right hand side lies within $F(R)$ since for any $f \in P$, $f = 0_A$ in A so $\phi(0_A) = \phi(f)$. For example, if $f = X_1X_4 - X_2X_3 - 1_A$ then

$$0_R = \phi(0_A) = \phi(X_1X_4 - X_2X_3 - 1_A) = \phi(X_1)\phi(X_4) - \phi(X_2)\phi(X_3) - 1_R$$

which is exactly the condition for $(\phi(X_i))_{i\in\{1,\ldots,4\}} \in F(R)$.

It is injective, since ϕ is determined by where it sends indeterminants, and is surjective because it has right inverse

$$\begin{aligned} F(R) &\to {}_k\mathbf{Alg}(A,R) \\ (x_i)_{i\in I} &\mapsto (X_i \mapsto x_i) \end{aligned} \tag{5.2}$$

where the right hand side makes sense exactly because (x_i) is a solution to every equation in P. □

We can see A as representing the most general solution to the equations in P. Note that I may not be finite, or even countable.

Our experience with categories so far should help us to notice $F(R)$ looks suspiciously like a functor. The fact that k-algebra homomorphisms preserve solutions to polynomial equations confirms our suspicions. So our claim has the somewhat grander statement:

Theorem 5.1.1. *Let $F\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ be a functor which finds the set of solutions in a k-algebra to $P \subset k[X_1,\ldots,X_n]$. Then there is a k-algebra A such that F and ${}_k\mathbf{Alg}(A,-)$ are naturally isomorphic.*

Proof of Naturality. Let $f\colon R \longrightarrow S$ be a k-algebra homomorphism. We want that the diagram from eq. (2.3)

$$\begin{array}{ccc} F(R) & \longrightarrow & {}_k\mathbf{Alg}(A,R) \\ \downarrow & & \downarrow \\ F(S) & \longrightarrow & {}_k\mathbf{Alg}(A,S) \end{array}$$

commutes. For a set $(x_i) \in F(X)$, if we follow the left and bottom arrows (x_i) gets taken to the map $X_i \mapsto (f(x_i)$. On the other hand, following the top and right arrows it goes to $(X_i \mapsto (x_i)) \circ f$. These are equal, and since the top and bottom arrows are bijections this is sufficient for naturality. □

Definition 5.1.2. Let $\mathbf{C}$ be a category. Let $F\colon \mathbf{C} \Rightarrow \mathbf{Set}$ be a functor. F is called *representable* if and only if there is some $A \in \mathrm{Ob}(\mathbf{C})$ such that F is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathbf{C}}(A,-)$. In this case, we say that A *represents* F.

We note a couple of nice facts about representability:

Proposition 5.1.3.

1. *If $E\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ assigns every R a single point. Then it is represented by k.*
2. *If $F_1, F_2\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ are represented by A_1 and A_2 respectively, then $F_1 \times_k F_2$, which takes R to $F_1(R) \times F_2(R)$, is represented by $A \otimes B$.*

Given the amount of work we spent proving similar things in chapter 4, we can't help but notice that this seems to be expressing commutativity of some functor with certain limits. The functor in question takes a k-algebra A to ${}_k\mathbf{Alg}(A,-)$ (a functor itself).

These are nice enough results so far, reducing the quite involved process of picking solutions to instead looking at homomorphisms (again, we're reminded of the categorical attempt to "stop looking inside"), but it doesn't quite address what we asked for at the beginning of the section. We want groups, rather than sets, of solutions.

Definition 5.1.4 (Affine Group Scheme 1)**.** *An affine group scheme over* k is a representable functor $F\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Grp}$.

When we say that F is representable, what we mean is that it is representable when composed with the forgetful functor $\mathbf{Grp} \Rightarrow \mathbf{Set}$. In other words, it is representable when we forget the group structure. Note that the natural isomorphism means if F is represented by A, ${}_k\mathbf{Alg}(A,R)$ inherits the group structure of $F(R)$, but that we also need to restrict our functions $F(R) \to F(S)$ to group homomorphisms. This all feels rather clumsy. It would be much nicer if this property was simply about the representing object A. This is a perfect place to use our category theory toolbox.

5.2 Reconsidering Groups

As we've done before, let us take a definition we are familiar with, that of a group, and begin to work with it categorically:

Definition 5.2.1. *A group is a set* Γ *together with three maps:*

$$\begin{aligned}\mathrm{mult}\colon \Gamma \times \Gamma &\to \Gamma\\ \mathrm{unit}\colon \{e\} &\to \Gamma\\ \mathrm{inv}\colon \Gamma &\to \Gamma\end{aligned}$$

such that the following diagrams commute:

$$\begin{tikzcd} \Gamma \times \Gamma \times \Gamma \arrow[r, "\mathbf{1}_\Gamma \times \mathrm{mult}"] \arrow[d, "\mathrm{mult} \times \mathbf{1}_\Gamma"'] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\ \Gamma \times \Gamma \arrow[r, "\mathrm{mult}"] & \Gamma \end{tikzcd} \tag{5.3}$$

for associativty;

$$\begin{tikzcd} \{e\} \times \Gamma \arrow[r, "\mathrm{unit} \times \mathbf{1}_\Gamma"] \arrow[d, phantom, "\wr\|"] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\ \Gamma \arrow[r, equal] & \Gamma \end{tikzcd} \tag{5.4}$$

for left unit and

$$\begin{tikzcd} \Gamma \arrow[r, "(\mathrm{inv}, \mathbf{1}_\Gamma)"] \arrow[d] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\ \{e\} \arrow[r, "\mathrm{unit}"] & \Gamma \end{tikzcd} \tag{5.5}$$

for left inverse.

Recall. Given morphisms $f_1\colon X \longrightarrow Y_1$ and $f_2\colon X \longrightarrow Y_2$ in a category with products, (f_1, f_2) is the unique map they induce $X \longrightarrow Y_1 \times Y_2$. On the other hand, assume that $f_1\colon X_1 \longrightarrow Y_1$ and $f_2\colon X_2 \longrightarrow Y_2$, then the morphism $f_1 \times f_2\colon X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ is the morphism induced by the arrows $f_i \circ \pi_i\colon X_1 \times X_2 \longrightarrow Y_i$. In **Set**, this is just $f_1 \times f_2(x_1, x_2) = (f(x_1), f(x_2))$.

So now let us imagine that we have a functor $F\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$. In this new approach to groups, what does it mean for F is induce a functor $\mathbf{C} \Rightarrow \mathbf{Grp}$? Well, for each k-algebra, R, we must choose a triple of maps $\mathrm{mult}_R, \mathrm{inv}_R$ and unit_R with the above commuting diagrams. Further, we need that for any map $f\colon R \longrightarrow S$, the diagram

$$\begin{tikzcd} F(R) \times F(R) \arrow[r, "\mathrm{mult}_R"] \arrow[d, "F(f) \times F(f)"'] & F(R) \arrow[d, "F(f)"] \\ F(S) \times F(S) \arrow[r, "\mathrm{mult}_S"] & F(S) \end{tikzcd}$$

commutes, and with similar diagrams for inv and unit. But this says exactly that mult is a natural transformation $F \times F \Rightarrow F$ which satisfies eq. (5.3) for every k-algebra!

So we have the new definition:

Definition 5.2.2 (Affine Groups Scheme 2)**.** *An affine group scheme over k is a representable functor $F\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ equipped with three natural transformations mult, inv and unit, satisfying eq. (5.3), eq. (5.4) and eq. (5.5) for every k-algebra R.*

This process is not quite over, though. We have started to unify our definition, but we can do one better.

5.3 Yoneda's Lemma and Hopf Algebras

We would like to understand what the presence of these natural transformations in definition 5.2.2 does to the representing object A, and whether we can ensure a representable functor ${}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ is an affine group scheme simply by picking A well. We are interested, then, in establishing a correspondence between natural transformations of representable functors and something else. It would be convenient if this correspondence depended only on the representing object, and even better if we could put it together in a categorical "don't look inside" manner. Our next result, then, is perfect:

Theorem 5.3.1 (Yoneda's Lemma)**.** *Let* $\mathbf{C}$ *be a locally-small category and let* $E, F\colon \mathbf{C} \Rightarrow \mathbf{Set}$ *be representable functors, represented by objects* A *and* B *respectively. Then there is a bijection between* $\mathrm{Nat}(E, F)$*, the set of natural transformations* $E \Rightarrow F$*, and* $\mathrm{Hom}_{\mathbf{C}}(B, A)$*.*

Proof. Let $\phi \in \mathrm{Hom}_{\mathbf{C}}(B, A)$. Now let $R \in \mathrm{Ob}\,(\mathbf{C})$. By representability, we have isomorphisms $E(R) \cong \mathrm{Hom}_{\mathbf{C}}(A, R)$ and $F(R) \cong \mathrm{Hom}_{\mathbf{C}}(B, R)$, so to get a natural transformation it is sufficient to provide a morphisms $\mathrm{Hom}_{\mathbf{C}}(A, R) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(B, R)$ which fulfil the naturality condition. For $\psi\colon A \longrightarrow R$, we take it to $\psi \circ \phi\colon B \longrightarrow R$.

Let us show that this is a natural map. Let $f\colon R \longrightarrow S$ be a morphism and recall that

$$\mathrm{Hom}_{\mathbf{C}}(A, f) = f \circ -.$$

Then we need commutativity of

$$\begin{tikzcd}
E(R) \cong \mathrm{Hom}_{\mathbf{C}}(A, R) \arrow[r, "- \circ \phi"] \arrow[d, "f \circ -"] & \mathrm{Hom}_{\mathbf{C}}(B, R) \cong F(R) \arrow[d, "f \circ -"] \\
E(S) \cong \mathrm{Hom}_{\mathbf{C}}(A, S) \arrow[r, "- \circ \phi"] & \mathrm{Hom}_{\mathbf{C}}(B, S) \cong F(S)
\end{tikzcd}$$

Following either branch we see that $\varphi \mapsto f \circ \varphi \circ \phi$ and this give us what we want.

Going the other direction, let $\Phi\colon E \Rightarrow F$ be a natural transformation. Then for any morphims $f\colon R \longrightarrow S$ we have the commutative diagram

$$\begin{tikzcd}
E(R) \cong \mathrm{Hom}_{\mathbf{C}}(A, R) \arrow[r, "\Phi_R"] \arrow[d, "f \circ -"] & \mathrm{Hom}_{\mathbf{C}}(B, R) \cong F(R) \arrow[d, "f \circ -"] \\
E(S) \cong \mathrm{Hom}_{\mathbf{C}}(A, S) \arrow[r, "\Phi_S"] & \mathrm{Hom}_{\mathbf{C}}(B, S) \cong F(S)
\end{tikzcd}$$

so we may take $R = A$ and consider where Φ_R takes $\mathbf{1}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ in $\mathrm{Hom}_{\mathbf{C}}(B, A)$ (along the top arrow). Let us call this $\phi\colon B \longrightarrow R$. Then following $\mathbf{1}_A$ around both sides of the diagram, we find $\Phi_S(f) = f \circ \phi$, so this is inverse to the operation above. $\square$

If we were to further explore the functor which takes A to $\mathrm{Hom}_{\mathbf{C}}(A, -)$, and the the categories involved, we would be able to show that this is actually a natural isomorphism.

Corollary 5.3.2. *If* Φ *and* ϕ *correspond to each other as above, then* Φ *is a natural isomorphism if and only if* ϕ *is an isomorphism.*

It is not hard to prove, that commutative diagrams are transferred by this correspondence, with a reversal of arrows.

This result is exactly what we hoped for. It allows us to translate our information about affine group schemes in terms of natural transformations into a definition entirely in terms of k-algebra homomorphisms. (If this general instance was a little hard to follow, it might be worth reading the proof through when specialised to k-algebras in appendix A.4).

Definition 5.3.3. *A Hopf algebra over a field* k is a k-algebra with k-algebra maps

comultiplication	$\Delta\colon$	$A \to A \otimes A$
counit (augmentation)	$\varepsilon\colon$	$A \to k$
coinverse (antipode)	$S\colon$	$A \to A$

such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\mathbf{1}_A \otimes \Delta"'] \\
A \otimes A \arrow[u, "\Delta \otimes \mathbf{1}_A"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}, \quad
\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes \mathbf{1}_A"'] \\
A \arrow[u, phantom, "\wr\|"] \arrow[r, equal] & A \arrow[u, "\Delta"']
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
A & A \otimes A \arrow[l, "{(S, \mathbf{1}_A)}"'] \\
k \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "\Delta"']
\end{tikzcd}$$

Remark. Unlike in this project, other authors may not assume commutativity when defining Hopf Algebras.

Recall. Given three k-algebras A, B and C, and k-algebra homomorphisms $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$, the map $(f, g)\colon A \otimes B \longrightarrow C$ takes $a \otimes b \mapsto f(a)g(b)$, whilst $f \otimes g\colon A \otimes B \longrightarrow C \otimes C$ takes $a \otimes b \mapsto f(a) \otimes g(b)$.

Using proposition 5.1.3, we see these are exactly the dual properties to the natural transformations of definition 5.2.2 and so we get the rather pleasing result:

Theorem 5.3.4. *Affine groups schemes over k correspond to Hopf algebras over k.*

Example. Let us take a moment to discuss a specific example. As we discussed in section 5.1, finding the multiplicative group of a k-algebra R is finding pairs $(x_1, x_2) \in R^2$ satisfying $x_1x_2 = 1_R$. So we can see that the required representing algebra is going to be

$$A = k[X_1, X_2]/\langle X_1X_2 - 1\rangle.$$

What Hopf algebra structure on A is appropriate here, to give us the functor F which takes R to R^*?

We know the natural transformation mult: $F \times F \Rightarrow F$ has $\mathrm{mult}_R\colon (r, s) \mapsto rs$. So let us following the method from our proof of theorem 5.3.1.

We are looking for the image of $\mathbf{1}_{A\otimes A}$ under the map

$$F(A \otimes A) \times F(A \otimes A) \cong {}_k\mathbf{Alg}(A \otimes A, A \otimes A) \xrightarrow{\text{mult}} {}_k\mathbf{Alg}(A, A \otimes A) \cong F(A \otimes A).$$

Observe that

$$A \otimes A \cong k[X_1 \otimes 1, X_2 \otimes 1, 1 \otimes X_1, 1 \otimes X_2]/\langle X_1X_2 - 1\rangle$$

where we are abusing the notation $\langle X_1X_2 - 1\rangle$ to mean

$$\langle (X_1X_2 - 1) \otimes 1, 1 \otimes (X_1X_2 - 1)\rangle$$

In $F(A\otimes A)\times F(A\otimes A)$ then, $\mathbf{1}_{A\otimes A}$ corresponds, by eq. (5.1), to the 4-tuple $(X_1\otimes 1, X_2\otimes 1, 1\otimes X_1, 1\otimes X_2)$ in $F(A\otimes A) \times F(A\otimes A)$. mult takes this to $(X_1 \otimes X_1, X_2 \otimes X_2)$ in $F(A \otimes A)$, which then corresponds to the map $(X_i \mapsto X_i \otimes X_i)$ by eq. (5.2). So comultiplication is exactly this.

$$\begin{aligned} \Delta\colon A &\to A \otimes A \\ X_i &\mapsto X_i \otimes X_i \end{aligned}$$

The second commutative diagram, tells us that $\varepsilon(X_i) \otimes X_i = 1 \otimes X_i$ so

$$\begin{aligned} \varepsilon\colon A &\to k \\ X_i &\mapsto 1 \end{aligned}$$

and the last diagram tells us that $S(X_i)X_i = 1$, so

$$\begin{aligned} S\colon A &\to A \\ X_i &\mapsto X_i^{-1} \end{aligned}$$

where $X_1^{-1} = X_2$ and $X_2^{-1} = X_1$.

So the affine group scheme which takes a k-algebra to its group of multiplicative units is represented by the Hopf algebra A with operations Δ, ε and S.

The theory of affine group schemes is rich in its own right. Due to the correspondences established in this chapter, it seems obvious to investigate how restricting the type of group scheme or the type of Hopf algebra effects the other object. It is not difficult to show that an affine group scheme is commutative if and only if comultiplication is symmetric under permutation of the two tensor factors (in this case, we say A is *cocommutative*). We can define affine group scheme homomorphisms as natural maps which act as group homomorphisms on each object, and Yoneda gives us that natural maps between affine group schemes have corresponding algebra homomorphisms, so what might these two things tell us about one another? Does it affect our affine group schemes to require that A is finitely generated?

Unfortunately, we must leave affine group schemes without exploring these, but [11] is a fantastic source for the interested reader. However, we do have one last set of observations to make.

5.4 Representations of an Affine Group Scheme

Definition 5.4.1. Let $G\colon \mathbf{C} \Rightarrow \mathbf{Grp}$ and $X\colon \mathbf{C} \Rightarrow \mathbf{Set}$ be functors. *An action of G on X* is a natural transformation $\alpha\colon G \times X \Rightarrow X$ such that for all $R \in \mathrm{Ob}\,(\mathbf{C})$, $\alpha_R\colon G(R) \times X(R) \longrightarrow X(R)$ is a group action.

Remark. Given two categories $\mathbf{C}$ and $\mathbf{D}$, we may produce the product category $\mathbf{C}\times\mathbf{D}$ for which objects are just pairs (C, D) for $C \in \mathrm{Ob}\,(\mathbf{C})$ and $D \in \mathrm{Ob}\,(\mathbf{D})$ and morphisms are pairs of arrows componentwise. So we may talk of the product of functors. For example, in definition 5.4.1, $G \times X\colon \mathbf{C} \Rightarrow \mathbf{Grp} \times \mathbf{Set}$ takes R to $G(R) \times X(R)$.

Definition 5.4.2. Let us fix a field k and a k-vector space V. Let $X\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ be the functor with $X(R) = V \otimes R$. An action $G(R) \times (V \otimes R) \longrightarrow V \otimes R$ is called R-linear if, for every $g \in G(R), v, v' \in V$ and $r, r' \in R$, we have

$$g(v \otimes r + v' \otimes r') = g(v) \otimes r + g(v') \otimes r'$$

If $\alpha\colon G \times X \Rightarrow X$ is an action of G on X and α_R is R-linear for all k-algebras R, then we say we have *a linear representation of G on V*.

When it is clear in the context, often the notation $v \otimes r$ is abandoned and instead rv is written.

As we've become used to, there is a Hopf algebra equivalent formulation here.

Theorem 5.4.3. *Let $G\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Grp}$ be an affine group scheme represented by a Hopf algebra A. The linear representations of G on V correspond to k-linear maps $\rho\colon V \longrightarrow V \otimes A$ such that*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes A \arrow[d, "\mathbf{1}_V \otimes \Delta"] \\ V \otimes A \arrow[r, "\rho \otimes \mathbf{1}_A"'] & V \otimes A \otimes A \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} V \arrow[r, "\sim"] \arrow[dr, "\rho"'] & V \otimes k \\ & V \otimes A \arrow[u, "\mathbf{1}_V \otimes \varepsilon"'] \end{tikzcd}$$

commute.

Though we will not give the proof here (it can be found in [11]), we will note that this should not be a surprise. In definition A.3.1 we have two conditions for a map $\sigma\colon \Gamma \times X \to X$ to be a group action and they can be expressed by saying

$$\begin{tikzcd} X & \Gamma \times X \arrow[l, "\mathrm{mult}"'] \\ \Gamma \times X \arrow[u, "\sigma"] & \Gamma \times \Gamma \times X \arrow[l, "\mathbf{1}_\Gamma \times \sigma"] \arrow[u, "\mathrm{mult} \times \mathbf{1}_X"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X & \{e\} \times X \arrow[l, "\sim"'] \arrow[d, "\mathrm{unit} \times \mathbf{1}_X"] \\ & \Gamma \times X \arrow[ul, "\sigma"] \end{tikzcd}$$

commute. The two diagrams in theorem 5.4.3 are dual to these.

Definition 5.4.4. A k-module V with a k-linear $\rho\colon V \longrightarrow V \otimes A$ satisfying $(\mathbf{1}_V \otimes \varepsilon) \circ \rho = \mathbf{1}_V$ and $(\mathbf{1}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathbf{1}_A) \circ \rho$ is called *an A-comodule*.

So theorem 5.4.3 tells us that linear representations are in correspondence with A-comodules. This result is important. It is by this correspondence that the equivalence of categories, the statement of which this last section is building to, is proved.

It also might allows us to avoid a slightly tricky situation. Given two linear representations of G, $G \times (V \otimes -) \Rightarrow V \otimes -$ and $G \times (W \otimes -) \Rightarrow W \otimes -$, we would like to define morphisms between them. Morphisms between categories are functors and morphisms between functors are natural transformations but morphisms between natural transformations have not been covered. The above theorem tells us that we can understand morphisms between linear representations by transferring morphisms between comodules and it is via this that we get to our definition:

Definition 5.4.5. Let $G\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Grp}$ be an affine group scheme. *The category, $\mathbf{Rep}_k(G)$, of linear representation of G* has as its objects linear representations of G on any finite-dimensional k-vector space V.

Morphisms between representations $G \times (V \otimes -) \Rightarrow V \otimes -$ and $G \times (W \otimes -) \Rightarrow W \otimes -$ are given by the k-linear maps f that make

$$\begin{tikzcd} G(R) \times (V \otimes R) \arrow[r] \arrow[d, "\mathbf{1}_{G(R)} \times (f \otimes \mathbf{1}_R)"'] & V \otimes R \arrow[d, "f \otimes \mathbf{1}_R"] \\ G(R) \times (W \otimes R) \arrow[r] & W \otimes R \end{tikzcd}$$

commute for all k-algebras R.

Chapter 6

Tannakian Categories

In our last chapter, we are going to step away from the meticulous, proof-based, example-informed mathematics we have been doing up until this point. The material is explored in great depth, with much more concern for lucidity, in [3] and this is largely what we will be following. Here, though, we have one aim: to define everything in the following definition.

Definition 6.0.1. [Neutral Tannakian Category] *A neutral Tannakian category over a field k* is a rigid abelian tensor category $(\mathbf{C}, \otimes)$ such that $k = \mathrm{End}(\mathbb{1})$ for which there exists an exact faithful k-linear tensor functor $\omega\colon \mathbf{C} \Rightarrow \mathbf{Vect}_k$. Any such functor is said called a *fibre functor*.

We already know what an exact, faithful functor is; everything else is new.

6.1 Tensor Categories

Tensor categories are a natural attempt to generalise the properties of categories with a tensor product. In the past, tensor products have arisen as the products or coproducts in particular other categories, but here, as we have become used to, we distil them down to a simple set of properties. As we will see, most of this is to do with rephrasing associativty and commutativity in terms of isomorphisms, but perhaps the most interesting quality we require, which separates the tensor product from other instances of products we have seen, is the existence of an identity object.

Definition 6.1.1. Let $\mathbf{C}$ be a category and let $\otimes\colon \mathbf{C} \times \mathbf{C} \Rightarrow \mathbf{C}$ be a functor. We write $X \otimes Y$ for $\otimes(X, Y)$.

An *associativity constraint for* $(\mathbf{C}, \otimes)$ is an collection of isomorphisms[1], for each $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$,

$$\phi_{X,Y,Z}\colon X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z\colon$$

for which the diagram, called *the pentagon axiom*,

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "\mathbf{1}_X \otimes \phi"'] \arrow[dr, "\phi"] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "\phi"] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d, "\phi"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi \otimes \mathbf{1}_T"] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}$$

commutes, where we omit the subscripts to avoid clutter.

A *commutativity constraint for* $(\mathbf{C}, \otimes)$ is a collection of isomorphisms, for each $X, Y \in \mathrm{Ob}(\mathbf{C})$,

$$\psi_{X,Y}\colon X \otimes Y \xrightarrow{\sim} Y \otimes X$$

[1] What we really mean here is that we have functors $F_1, F_2\colon \mathbf{C} \times \mathbf{C} \times \mathbf{C} \Rightarrow \mathbf{C}$ which take (X, Y, Z) to $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$ respectively, and $\phi\colon F_1 \Rightarrow F_2$ is a natural isomorphism between them. Similarly for the commutativity constraint where we are looking for a natural isomorphism between the functors that take (X, Y) to $X \otimes Y$ and $Y \otimes X$.

such that

$$\psi_{Y,X} \circ \psi_{X,Y} = \mathbf{1}_{X \otimes Y} \colon X \otimes Y \Rightarrow X \otimes Y.$$

An associativity constraint and a commutativity constraint are called *compatible*, if and only if for all objects $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$, the diagram, called *the hexagon axiom*,

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\phi"] \arrow[dl, "\mathbf{1}_X \otimes \psi"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi"'] & & & Z \otimes (X \otimes Y) \arrow[dl, "\phi"] \\
 & (X \otimes Z) \otimes Y \arrow[r, "\psi \otimes \mathbf{1}_Y"] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

commutes.

An identity object of $(\mathbf{C}, \otimes)$ is a pair (U, u) for $U \in \mathrm{Ob}(\mathbf{C})$ and an isomorphism $u \colon U \xrightarrow{\sim} U \otimes U$ such that the functor $\mathbf{C} \longrightarrow \mathbf{C}$ which takes $X \mapsto U \otimes X$ is an equivalence of categories.

A tensor category is a system $(\mathbf{C}, \otimes, \phi, \psi)$ where ϕ and ψ are compatible associativity and commutativity constraints, which has an identity object.

Given a tensor category, it is not difficult to show that any identity object is unique up to unique isomorphism, so we may talk of *the* identity object, and denote it/one by $(\mathbb{1}, e)$. We will also assume all tensor categories to be locally small in our discussion.

6.2 Abelian Tensor Categories

We are already familiar with a number of tensor categories. Categories of R-modules for any ring R, or k-algebras, form tensor categories with the standard tensor products. So does **Set** with the Cartesian products (identity objects are just singletons). The category **AbGrp** is particularly interesting, because it comes with many added layers of structure that may be of use to us.

The first interesting property is that the trivial group $\{e\}$ is both initial and final in **AbGrp**.

Definition 6.2.1. Let $\mathbf{C}$ be a category. An object $Z \in \mathrm{Ob}(\mathbf{C})$ which is both initial and final is called *a zero object.*

If $\mathbf{C}$ has a zero object, 0, then for any $X, Y \in \mathrm{Ob}(\mathbf{C})$, there is exactly one morphism through 0, $0_{XY} \colon X \longrightarrow 0 \longrightarrow Y$, and we call this *a zero morphism.* There are ways to discuss these in the absence of zero objects, but that's not necessary here.

Zero objects in some sense exist in a canonical way inside every object in the category, and every object can be mapped into them. We have many familiar examples: $\{0\}$ in **Grp**, R-**Mod** and $\mathbf{Vect}_k$, the space of just one point in $\mathbf{Top}_*$. They allow us to expand to categories the idea of killing parts of an algebraic object by mapping them to zero:

Definition 6.2.2. Let $\mathbf{C}$ be a category with a zero object, 0. Let $f \colon X \longrightarrow Y$ be a morphism. If it exists, *the kernel of* f, $\ker f$, is the equaliser of f and the zero morphism 0_{XY}.

In all our familiar categories, this conforms with exactly what we'd expect. We also have the co-construction.

Definition 6.2.3. Let $\mathbf{C}$ be a category with a zero object, 0. Let $f \colon X \longrightarrow Y$ be a morphism. If it exists, *the cokernel of* f, $\mathrm{coker}(f)$, is the coequaliser of f and the zero morphism 0_{XY}.

In **Grp**, this is the projection onto the quotient of Y by the normal closure of $/\mathrm{im}(f)$.

It is a reasonable question to ask which morphisms have kernels and cokernels (in **Grp**, all morphisms do) and which objects appear as the kernel or cokernel of morphisms.

Definition 6.2.4. A monomorphism $f \colon E \longrightarrow X$ in a category $\mathbf{C}$ with kernels is called *normal* if and only if it is the kernel of some morphism $\phi \colon X \longrightarrow Y$.

An epimorphism $g \colon Y \longrightarrow A$ in a category $\mathbf{C}$ with cokernels is called *normal* if and only if it is the cokernel of some morphism $\phi \colon X \longrightarrow Y$

This separates **Grp** from **AbGrp**:

Proposition 6.2.5. *An injective group homomorphism $f\colon H \longrightarrow G$ is normal if and only if* $\mathrm{im}(f)$ *is a normal subgroup of G.*

Proof. If $\mathrm{im}(f)$ is normal then f is the kernel of the projection $G \to G/\mathrm{im}(f)$. On the other hand, the kernel of any group homomorphism is a normal subgroup, so if $\mathrm{im}(f)$ is not, then f is not normal. □

So we see that every monomorphism in **AbGrp** is normal, whilst this is not true in **Grp**. On the other hand, note that if a group homomorphism $g\colon Y \longrightarrow A$ is surjective, then the inclusion $\ker g \hookrightarrow Y$, has cokernel $Y/\ker g$, and $Y/\ker g \cong \mathrm{im} g = A$. So every epimorphism in **Grp** is normal.

Our last observation is that in **AbGrp**, products and coproducts are the same objects.

Definition 6.2.6. In a category $\mathbf{C}$, an object $X_1 \oplus \ldots X_n$ is called *a biproduct of* $\{X_i\}$ if and only if it has projections and inclusions that make it into both a product and a coproduct of the collection $\{X_i\}$.

This is not true in **Grp**, where the coproduct is the free product.

Definition 6.2.7 (Abelian categories)**.** *An abelian category* is a locally-small category $\mathbf{C}$ which has a zero object and all finite biproducts. Further, every kernel and cokernel exists and all monic and epic morphisms are normal.

It can be shown that these conditions give the Hom sets of $\mathbf{C}$ the structure of additive abelian groups and that morphism composition is bi-additive. This is a particularly nice property of **AbGrp** and it is nice to see it arise here.

Definition 6.2.8 (Abelian tensor categories)**.** A functor between abelian categories is called *additive* if it commutes with zero objects and finite biproducts.

An abelian tensor category is a tensor category $(\mathbf{C}, \otimes)$ such that $\mathbf{C}$ is an abelian category and $\otimes$ is a bi-additive functor.

Note that if $(\mathbf{C}, \otimes)$ is an abelian tensor category with identity object $(\mathbb{1}, e)$, this group structure on Hom sets means that $\mathrm{End}(\mathbb{1}) := \mathrm{Hom}_{\mathbf{C}}(\mathbb{1}, \mathbb{1})$ is in fact a ring.

6.3 Rigid Tensor Categories

In the category of R-modules, there is a well established isomorphism for any R-modules X, Y and T,

$$\mathrm{Hom}_R(T, \mathrm{Hom}_R(X, Y)) \cong \mathrm{Hom}_R(T \otimes_R X, Y)$$

which tells us exactly that $\mathrm{Hom}_R(X, Y)$ represents the contravariant functor $R\text{-mod}^{\mathrm{op}} \Rightarrow \mathbf{Set}$ which takes T to $\mathrm{Hom}_R(T \otimes_R X, Y)$.

Definition 6.3.1. Let $(\mathbf{C}, \otimes)$ be a tensor category. Let us introduce the functor $\mathbf{C}^{\mathrm{op}} \Rightarrow \mathbf{Set}$, for two fixed $X, Y \in \mathrm{Ob}(\mathbf{C})$, which takes T to $\mathrm{Hom}_R(T \otimes X, Y)$. If this functor is representable, then we denote the representing object by $\underline{\mathrm{Hom}}(X, Y)$. It is called *the internal Hom from X to Y*.

Then, since $\mathrm{Hom}_{\mathbf{C}}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y)$ is the image of $\underline{\mathrm{Hom}}(X, Y)$ under the functor, representability gives an bijection $\mathrm{Hom}_{\mathbf{C}}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$.

Definition 6.3.2. *The evaluation map* is the morphism $\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}(X, Y) \otimes X \longrightarrow Y$ corresponding to $\mathbf{1}_{\underline{\mathrm{Hom}}(X,Y)}$.

Internal Homs generalise the many instances we have of Hom sets have the structure of objects, or objects behaving like a Hom sets. In for R-modules, the evaluation map takes $f \otimes x$ to $f(x)$, as we'd expect.

Note that for $X_1, X_2, Y_1, Y_2 \in \mathrm{Ob}(\mathbf{C})$, we may make the map

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \otimes (X_1 \otimes X_2) \longrightarrow Y_1 \otimes Y_2$$

corresponding to $\mathrm{ev}_{X_1,Y_1} \otimes \mathrm{ev}_{X_2,Y_2}$ (with a little associativity and commutativity), and then use representability to yield a corresponding morphism

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \longrightarrow \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2). \tag{6.1}$$

An internal Hom allows us to also generalise the idea of the dual of a vector space.

Definition 6.3.3. Let $(\mathbf{C}, \otimes)$ be a tensor category. Let $X \in \mathrm{Ob}(\mathbf{C})$. If it exists, *the dual of* X, denoted X^*, is defined to be $\underline{\mathrm{Hom}}(X, \mathbb{1})$. This comes with the evaluation map $\mathrm{ev}_X : X^* \otimes X \longrightarrow \mathbb{1}$.

Exactly by the definition of the internal Hom, we have a collection of isomorphisms (in fact, a natural isomorphism, if all duals do exist)

$$\mathrm{Hom}_{\mathbf{C}}(T \otimes X, \mathbb{1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(T, \underline{\mathrm{Hom}}(X, \mathbb{1})) = \mathrm{Hom}_{\mathbf{C}}(T, X^*)$$

Then, if X^{**} exists, this includes an isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(X \otimes X^*, \mathbb{1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(X, X^{**})$$

and so the morphism $\mathrm{ev}_X \circ \psi : X \otimes X^* \longrightarrow \mathbb{1}$ corresponds to a morphism $i_X : X \longrightarrow X^{**}$.

Definition 6.3.4. An object X in a tensor category $(\mathbf{C}, \otimes)$ is called *reflexive* if X^{**} exists and the map i_X is an isomorphism.

Definition 6.3.5 (Rigid tensor categories)**.** Let $(C, \otimes)$ be a tensor category. It is said to be *rigid* if and only if:

1. For any $X, Y \in \mathrm{Ob}(\mathbf{C})$, $\underline{\mathrm{Hom}}(X, Y)$ exists.
2. For any objects $X_1, X_2, Y_1, Y_2 \in \mathrm{Ob}(\mathbf{C})$, the morphism from eq. (6.1), is an isomorphisms.
3. All objects of $\mathbf{C}$ are reflexive.

6.4 Tensor Functors

We would like functors between tensor categories to preserve the structure in some sense:

Definition 6.4.1 (Tensor functor)**.** Let $(\mathbf{C}, \otimes)$ and $(\mathbf{C}', \otimes')$ be tensor categories. *A tensor functor* is a pair (F, c) where $F : \mathbf{C} \Rightarrow \mathbf{C}'$ is functor and c is a collection of isomorphisms[2], for each $X, Y \in \mathrm{Ob}(\mathbf{C})$

$$c_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

with the following properties:

1. For any $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$, we have the commutative diagram

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "\mathbf{1}_{FX} \otimes c"] \arrow[d, "\phi'"] & FX \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi)"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "c \otimes \mathbf{1}_{FZ}"] & F(X \otimes Y) \otimes FZ \arrow[r, "c"] & F((X \otimes Y) \otimes Z).
\end{tikzcd}$$

2. For any $X, Y \in \mathrm{Ob}(\mathbf{C})$, we have the commutative diagram

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "c"] \arrow[d, "\psi'"] & F(X \otimes Y) \arrow[d, "F(\psi)"] \\
FY \otimes FX \arrow[r, "c"] & F(Y \otimes X).
\end{tikzcd}$$

3. If (U, u) is an identity object of $\mathbf{C}$, then $(F(U), F(u))$ is an identity object of $\mathbf{C}'$.

Definition 6.4.2 (Tensor equivalence)**.** *A tensor equivalence*, or *equivalence of tensor categories*, is a tensor functor $(F, c) : (C, \otimes) \Rightarrow (C', \otimes')$ such that $F : \mathbf{C} \Rightarrow \mathbf{C}'$ is an equivalence of categories.

[2]Again, this is actually a natural isomorphism between the functors $\mathbf{C} \times \mathbf{C} \Rightarrow \mathbf{C}'$ which send $X, Y \in \mathrm{Ob}(\mathbf{C})$ to $F(X) \otimes F(Y)$ and $F(X \otimes Y)$ respectively.

6.5 Neutral Tannakian Categories and a Final Equivalence

So we can now restate the definition that began this chapter, have now defined all the terms in it:

Definition 6.0.1. [Neutral Tannakian Category] *A neutral Tannakian category over a field k* is a rigid abelian tensor category $(\mathbf{C}, \otimes)$ such that $k = \mathrm{End}(\mathbb{1})$ for which there exists an exact faithful k-linear tensor functor $\omega \colon \mathbf{C} \Rightarrow \mathbf{Vect}_k$. Any such functor is said called a *fibre functor*.

With this now at hand, we can conclude our study with the statement of one last theorem. It can be found, with proof, as theorem 2.11 in [3]. In the same way as in our study of Galois categories, it is an equivalence between categories of algebraic objects and a category defined by highly abstract categorical constraints. If we are at risk of missing the enormity of this result because of how concisely it may be stated, let us not forget the vast number of technical definitions and theorems from disparate areas of maths that we have built just so that we might comprehend the statement:

Theorem 6.5.1. *Let $(\mathbf{C}, \otimes)$ be a neutral Tannakian category over a field k. Further, let $\omega \colon \mathbf{C} \Rightarrow \mathbf{Vect}_k$ be a fixed fibre functor. Then ω gives us a functor $\mathbf{C} \Rightarrow \mathbf{Rep}_k(G)$ for some affine group scheme G and this functor is an equivalence of tensor categories.*

Conclusion

First and foremost, this project is an attempt to lay out an example of a categorical approach to mathematics. At first, we did this in specific reference to the similarities between theorem 0.0.1 and theorem 0.0.2. With this motivation, we built the basic machinery of category theory, with particular emphasis on how this began to unite familiar concepts from throughout maths. This part of the project was completed with theorem 4.5.4, the main theorem of Galois categories. This result was the first that we could claim to be truly categorical in nature. It is a striking piece of evidence that the dictionary-building exercise we are undertaking has the capability to reveal deep and wide reaching results. Additionally, it opens the way to the study of fundamental groups in algebraic geometry. It by far eclipses the two results that inspired us to search for it and opens up the possibility that other algebraic relationships may be distilled down to purely categorical ones. The rest of the project was spent setting up the background to establish another one of these equivalences, this time between more complicated categories.

The most impressive aspect of these results is that the list of conditions in the definitions of Galois and Tannakian categories are so specific. The decomposition of morphisms into epic and monic morphisms in definition 4.1.1, for example, is an aspect of $\mathbf{Cov}_X$ and ${}_k\mathbf{SAlg}$ that seems inconsequential, or at least besides the point. When those properties are used, this is seen as a result of other, important properties of algebras or covering spaces, rather than due to some value of their own. The power of our approach here, then, is the ability to drill down to the most basic of parts what makes the proofs work and what makes these categories behave in this way. Fundamentally, category theory, as with much of maths, is about creating connections to bridge gaps of understanding. In our results here, we have shown that not only can category theory do this, but it can do it in ways no other areas can. Our concluding theorem tells us that we are only just beginning to understand what these connections may be.

We touch on many areas of maths and there are plenty of options for an interested reader to explore further. For pure category theory, the classic text is Mac Lane [7]. For a more modern text, Riehl [8] extends far beyond the scope of this project with a similar emphasis on clarity and examples. For more information specifically on Galois categories, Zomervrucht [12] is a good, if brief, set of notes, whilst Lenstra [5] is a much more extensive look into the theory, going on to study the Galois theory of schemes. Both sources have exercises. On the pure theory of affine group schemes, Waterhouse [11] is clear and insightful. Finally, Deligne and Milne [3] is the definitive text on Tannakian categories, both adding a lot of theory to the definitions given here and elaborating to a much greater extent.

Appendix A

Appendix

A.1 Separable algebras

The statement of theorems in this section closely follow chapter 8 of [6]. As in the source, the aim here to only to set up the main results necessary to give a key example of a Galois category, and so these will largely be stated without proof. Further information can be found in the source, with the proofs in chapter 2 of [5], as cited by [6].

In all that follows, we will being using k to denote a field. A lot of the content can be adapted for k to instead be a more general ring but this will be outside of the scope of this paper. This simplification is the biggest change from the source material. We will also assume all rings have 1 and are commutative. In particular, we are only considering commutative algebras.

Definition A.1.1. *A k-algebra* is a ring A equipped with a ring homomorphism $f\colon k \to A$ such that $f(1_k) = 1_A$ and the image of f is contained in the centre of A. f may be referred to as the *structure map.*

Recall. The centre of a ring A is the set $Z(A) = \{r \in A \mid \forall s \in A, rs = sr\}$

We may consider a k-algebra as a ring that contains k, or alternatively, a k-vector space equipped with a associative bilinear product and an identity element. More general definitions of algebras are used elsewhere where some of these restrictions are relaxed. Note that we may abuse notation here and talk about multiplying a element $r \in k$ and $a \in A$, notated ra, when in fact we mean $f(r)a$.

Definition A.1.2. For two k-algebras A and B, *a k-algebra homomorphism from A to B* is a ring homomorphism $\phi\colon A \to B$ such that for $r \in k$ and $a \in A$, $\phi(ra) = r\phi(a)$. As we'd like, this can be expressed by saying the following diagram commutes in **Ring**:

$$\begin{tikzcd} & k \arrow[dl, "f_A"'] \arrow[dr, "f_B"] & \\ A \arrow[rr, "\phi"] & & B \end{tikzcd} .$$

Such triangles are very familiar to us, looking suspiciously similar to the dual of those found in $\mathbf{Cov}_X$. Indeed the category of k-algebras ${}_k\mathbf{Alg}$ is a subcategory of the coslice category $k/\mathbf{Ring}$. A further observation is that the category of commutative algebras is exactly $k/\mathbf{CRing}$, where **CRing** is the category of commutative rings. When we are looking out for them, familiar categorical constructions appear everywhere.

Definition A.1.3. Suppose B is a k-algebra that is finite-dimensional as a k-vector space.Then, for $b \in B$, we may define the map

$$\begin{aligned} m_b\colon B &\to B \\ x &\mapsto bx. \end{aligned}$$

This is k-linear and so we define the trace of the element b as $\mathrm{Tr}(b) := \mathrm{Tr}(m_b)$. This map too is k-linear. Note too that $\mathrm{Tr}(r) = \dim(B) \cdot r$ for any $r \in k$.

$\mathrm{Hom}_{{}_k\mathbf{Alg}}(B, k)$ is itself a k-vector space and has the same dimension as B. We define a map

$$\begin{aligned} \phi\colon B &\to \mathrm{Hom}_{{}_k\mathbf{Alg}}(B, k) \\ x &\mapsto \phi_x \end{aligned}$$

Where $\phi_x : y \mapsto \mathrm{Tr}(xy)$. If ϕ is an isomorphism, then we say that B *is separable over* k.

Definition A.1.4. A polynomial $f \in k[X] \backslash \{0\}$ is called *separable* if it has no repeated roots in $\bar{k}$, an algebraic closure of k.

An element $\alpha \in \bar{k}$ is *separable* if its minimal polynomial f_k^α over k is separable.

If we have a field L with $k \subset L \subset \bar{k}$, we say L *is separable over* k if every $\alpha \in L$ is separable over k.

We say that L is *normal over* k if the minimal polynomial f_k^α over k for any $\alpha \in L$ factors into linear factors in $L[X]$.

Theorem A.1.5. *Let $\bar{k}$ be the algebraic closure of a field k. Let B be a k-algebra. Define $\bar{B} = B \otimes_k \bar{k}$, a $\bar{k}$-algebra.*

Then the following statements are equivalent:

1. *B is separable over k.*
2. *$\bar{B}$ is separable over $\bar{k}$.*
3. *There is some $n \geqslant 0$ such that $\bar{B} \cong \bar{k}^n$ as $\bar{k}$-algebras.*
4. *There exists a finite set $\{B_i\}_{i=1}^t$ of finite separable field extensions of k such that $B \cong \prod_{i=1}^t B_i$.*

A.2 Galois Field Theory

As in appendix A.1, we closely follow chapter 8 of [6]. We only to set up the main results without proof. The proofs may be found in chapter 2 of [5].

Definition A.2.1. A field extension L/k is *algebraic* if every element of L is a root of some polynomial in $k[X]$.

The group of automorphisms of a field L, $\mathrm{Aut}(L)$, is the set of isomorphisms $f : L \xrightarrow{\sim} L$, with the group operation of composition, Given a subgroup $G \leqslant \mathrm{Aut}(L)$, we may consider $L^G := \{x \in L \mid \sigma(x) = x \text{ for all } \sigma \in G\}$.

A field extension L/k is called *Galois* if and only if it is algebraic, and that there exists $G \leqslant \mathrm{Aut}(L)$, which fixes exactly k: $L^G = k$.

We define the *Galois group of the extension L/k* as

$$\mathrm{Gal}(L/k) = \mathrm{Aut}_k(L) := \{\phi \in \mathrm{Aut}(L) \mid \phi(x) = x \text{ for all } x \in k\}$$

Definition A.2.2. Let k be a field and let $\bar{k}$ be an algebraic closure of k. Let $F \subset k[X] \backslash \{0\}$ be a collection of non-zero polynomials.

The splitting field of F over k is the subfield of $\bar{k}$ generated by k and the roots in $\bar{k}$ of all the elements of F.

Theorem A.2.3. *Let k be a field and L a field extension with $k \subset L \subset \bar{k}$. Let*

$$I := \{E \subset L \mid E \textit{ is a finite Galois extension of } k\}.$$

I is a directed, partially ordered set under inclusion. Then the following statements are equivalent:

1. *The extension L/k is a Galois.*
2. *L is normal and separable over k.*
3. *There exists a set F of non-zero separable polynomials in k such that L is the splitting field of F over k.*
4. *$\cup_{E \in I} E = L$.*

Further, if any of these conditions are satisfied, then

$$\mathrm{Gal}(L/k) \cong \varprojlim \mathrm{Gal}(E/k)$$

*where this is the limit of the diagram $I \Rightarrow$ **Grp** which takes E to* $\mathrm{Gal}(E/k)$.

Remark. We give $\mathrm{Gal}(L/k)$ its topology precisely by making $\mathrm{Gal}(E/k)$ a discrete space and letting this isomorphisms induce the corresponding profinite topology. This makes the isomorphism into an isomorphism in the category of topological groups.

Theorem A.2.4 (Main Theorem of Galois Theory)**.** *Let $k \subset L$ be a Galois extension of fields with Galois group $G := \mathrm{Gal}(L/k)$. We define the maps*

$$\begin{aligned}\phi\colon \{E \subset L \mid E \text{ is a field extension of } k\} &\to \{H \leqslant G \mid H \text{ is a closed subgroup}\}\\ E &\mapsto \mathrm{Aut}_E(L)\end{aligned}$$

and

$$\begin{aligned}\varphi\colon \{H \leqslant G \mid H \text{ is a closed subgroup}\} &\to \{E \subset L \mid E \text{ is a field extension of } k\}\\ H &\mapsto L^H.\end{aligned}$$

Then these maps are bijective and inverse to each other, reversing inclusions.

Further if $\phi(E) = H$ then we have

- *E/k is a finite extension if and only if H is open.*
- *If L/E is Galois then $\mathrm{Gal}(L/E) \cong H$.*
- *For all $\sigma \in G$, $\phi(\sigma[E]) = \sigma H \sigma^{-1}$.*
- *E/k is Galois if and only if H is normal in G. If this is true, then $\mathrm{Gal}(E/k) \cong G/H$.*

Definition A.2.5. *The separable closure of a field k* is defined to be

$$k_s := \left\{x \in \bar{k} \mid x \text{ is separable over } k\right\}.$$

The absolute Galois group of k is $\mathrm{Gal}(k_s/k)$.

A.3 Group Actions

Definition A.3.1. [9] If A is a set and Γ is a group, then A is called a *Γ-set* if it has a function $\sigma\colon \Gamma \times A \to A$, called *a group action*, denoted by $\sigma(g, a) = ga$, such that

1. For every $a \in A$, $1a = a$.
2. For every $g, h \in \Gamma$ and $a \in A$, $(gh)a = g(ha)$.

We may say in this case that Γ *acts on A*.

σ is called *transitive* if and only if for every $a, b \in A$, there is a $g \in \Gamma$ with $ga = b$.

σ is called *free* if and only if for $a \in A$ and $g, h \in \Gamma$, $ga = ha$ implies $g = h$.

In the case that A is a topological space, and in particular if it is a finite set and thus canonically a discrete space, and Γ is a topological group, we may meaningfully talk about the action σ being continuous. In such a case, A is called *a Γ-space*. Where it is clear that Γ is a topological group, we may still refer to Γ-spaces as Γ-sets.

Definition A.3.2. If we have two Γ-sets, A and B, *a morphism of Γ-sets from A to B* is a function $f\colon A \to B$ such that $f(ga) = gf(a)$ for all $g \in \Gamma$ and $a \in A$. On Γ-spaces, we require these to be continuous. In this way, we define the category of Γ-sets.

A.4 Yoneda for Separable Algebras

Theorem A.4.1 (Yoneda's Lemma for Separable Algebras)**.** *Let $E, F\colon {}_k\mathbf{Alg} \Rightarrow \mathbf{Set}$ be representable functors, represented by k-algebras A and B respectively. Then there is a bijection between $\mathrm{Nat}(E, F)$, the set of natural transformations $E \Rightarrow F$, and ${}_k\mathbf{Alg}(B, A)$.*

Proof. Let $\phi\colon B \longrightarrow A$. Now let R be a k-algebra. By representability, we have isomorphisms $E(R) \cong {}_k\mathbf{Alg}(A, R)$ and $F(R) \cong {}_k\mathbf{Alg}(B, R)$, so to get a natural transformation it is sufficient to provide a morphisms ${}_k\mathbf{Alg}(A, R) \longrightarrow {}_k\mathbf{Alg}(B, R)$ which fulfil the naturality condition. For $\psi\colon A \longrightarrow R$, we take it to $\psi \circ \phi\colon B \longrightarrow R$.

Let us show that this is a natural map. Let $f\colon R \longrightarrow S$ be a map of k-algebras and recall that ${}_k\mathbf{Alg}(A, f) = f \circ -$. Then we need commutativity of

$$\begin{array}{ccc} E(R) \cong {}_k\mathbf{Alg}(A,R) & \xrightarrow{-\circ\phi} & {}_k\mathbf{Alg}(B,R) \cong F(R) \\ \big\downarrow{\scriptstyle f\circ -} & & \big\downarrow{\scriptstyle f\circ -} \\ E(S) \cong {}_k\mathbf{Alg}(A,S) & \xrightarrow{-\circ\phi} & {}_k\mathbf{Alg}(B,S) \cong F(S) \end{array}$$

Following either branch we see that $\varphi \mapsto f \circ \varphi \circ \phi$ and this give us what we want.

Going the other direction, let $\Phi\colon E \Rightarrow F$ be a natural transformation. Then for any k-algebra homomorphims $f\colon R \longrightarrow S$ we have the commutative diagram

$$\begin{array}{ccc} E(R) \cong {}_k\mathbf{Alg}(A,R) & \xrightarrow{\Phi_R} & {}_k\mathbf{Alg}(B,R) \cong F(R) \\ \big\downarrow{\scriptstyle f\circ -} & & \big\downarrow{\scriptstyle f\circ -} \\ E(S) \cong {}_k\mathbf{Alg}(A,S) & \xrightarrow{\Phi_S} & {}_k\mathbf{Alg}(B,S) \cong F(S) \end{array}$$

so we may take $R = A$ and consider where Φ_R takes $\mathbf{1}_A$ in ${}_k\mathbf{Alg}(B, A)$ (along the top arrow). Let us call this $\phi\colon B \longrightarrow R$. Then following $\mathbf{1}_A$ around both sides of the diagram, we find $\Phi_S(f) = f \circ \phi$, so this is inverse to the operation above. $\square$

Bibliography

[1] Bradley, Tai-Danae. Limits and Colimits, Part 1 (Introduction) [Internet]. Math3ma. 2018 [cited 1 June 2019]. Available from https://www.math3ma.com/blog/limits-and-colimits-part-1.

[2] Goedecke, Julia. Category Theory [unpublished lecture notes, internet]. Part III Category Theory, University of Cambridge; presented Michaelmas 2013.
Available from: https://www.dpmms.cam.ac.uk/ jg352/pdf/CategoryTheoryNotes.pdf

[3] Deligne, Pierre. Milne, James S., *Tannakian Categories, in Hodge Cycles, Motives, and Shimura Varieties*, LNM 900, 1982, pp. 101-228.

[4] Hatcher, Allen. *Algebraic Topology.* Cambridge: Cambridge University Press; 2016

[5] Lenstra, H.W. Jr. Galois Theory for Schemes [unpublished lecture notes, internet] 3rd ed. Universiteit Leiden; 2008.
Available from: https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf

[6] Lynn, Melissa. *Galois Categories.* University of Chicago VIGRE REU thesis. University of Chicago; 2009.

[7] Mac Lane, Saunders. *Categories for the Working Mathematician* 2nd ed. New York: Springer; 1998.

[8] Riehl, Emily. *Category Theory in Context.* Mineola, New York: Dover publications; 2016.

[9] Rotman, Joseph J. *An Introduction to the Theory of Groups.* New York: Springer; 1995.

[10] de Vries, Sjoerd. Galois Categories [M3R thesis]. London: Imperial College London; 2018.

[11] Waterhouse, William C.. *Introduction to Affine Group Schemes.* Graduate texts in mathematics; 66. New York: Springer; 1979.

[12] Zomervrucht, Wouter. Fundamental Groups [unpublished lecture notes, internet]. Freie Universität Berlin; 2015.
Available from: http://www.math.leidenuniv.nl/ wzomervr/docs/fundamental_groups.pdf