Exchangeability Theorems as Categorical Limits of Probabilistic and Quantum Processes

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Abstract

The topic of this thesis is a sequence of results which re-situate exchangeability theorems from probability theory in categories of probabilistic and quantum processes. These theorems take sequences of events, say random variables, random matrices, or quantum states, and study what behaviours are possible after imposing invariance of the action of finite permutations of the elements of the sequence. These are often philosophically motivated, in the classical case by subjectivist foundations of probability, in the quantum case by the school of quantum Bayesianism. In chapters 3 and 4, quantum and classical de Finetti theorems are generalised to limits of diagrams in appropriate categories, moving from statements about measures and quantum states to theorems about parameterised random measures and quantum states. The latter also considers the use of multisets for encoding exchangeability. In chapter 5, an approach using coalgebras of functors as theories of systems is taken. Instead of looking at sequences of events, we consider processes, coalgebras, that take a parameter and probabilistically return both an output and an updated parameter. Such coalgebras can be exchangeable, and it is shown that these coalgebras have a universal object similar to in the limit objects of the previous chapters, namely a final exchangeable coalgebra. Both the multiset and coalgebraic theorems are given quantum analogues.

You wanna know how to rhyme?
You better learn how to add.
— Yasiin Bey

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[1] For this, I have Prof. Heather Harrington and Prof. Bob Coecke to thank.

Thank you to Steve S. on Math Stack Exchange for resolving a fundamental misunderstanding about C*-algebras within minutes of my publishing the question[2].

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I arrived in Oxford in October 2020 and in the first week met more people than I had in the proceeding six months, alongside beginning a research degree in *theoretical computer science*, which meant little to me at the time, and having to move flat because of a cockroach infestation. Half of that year was spent in London playing too much online chess and learning about sheaves, and when it came to an end I owed what remained of my mental health to Nicolas Garraud, Paul Micoud, Brianna Yang and Lindiwe Makgalemele.

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[2] Steve S (https://math.stackexchange.com/users/1557600/steve-s). *The Algebraic Tensor Product of C*-Algebras as an Operator System under Different Norms.* Mathematics Stack Exchange. Feb. 2025. eprint: `https://math.stackexchange.com/q/5031395`.

Roland, thank you for feeding me, in all senses of the word, in a time when I would have struggled to feed myself.

Another source of joy in this period was the radio show I cohosted with Ben Caspi, *Ben Nedis: The Tallest Mountain In Variety Town*, on the Oxide radio station[3]. Once a week we would chat and try to provide 'genre whiplash' for our listeners, of which there were few. It was really, really good fun, and to Ben, I am grateful for your company and your music.

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[3] For as long as it remains hosted online by Oxide, you may find the final episode here: https://www.nedsummers.com/BenNedisFinalEp

[4] Not to be mistaken for Clovis, the human man, earlier thanked in these acknowledgements

one month before when I intended to finish this degree (an insane decision on my part). CNM indeed.

I moved to Berlin with the hope of building a creative community. I have read poems and told stories on stage and in smoking areas around the city. Berlin is also blessed with bookshops and cinemas and I have read and watched more than any other year of my life, but far beyond these exercises in performance and consumption, the greatest creative addition to my Berlin has been Sara Bresciani. Sara, here we stand at the perfect moment of our lives to have met each other. It has been my privilege to be part of your Berlin, to have you be part of mine and to be able to count on your company in whatever follows.

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[5] Not to be mistaken for Stanley, the human man, earlier thanked in these acknowledgements.

[6] Luke Summers. ‘Echoes of the Shoah: British Jewry and the Bosnian War’. In: *Holocaust Studies* (Aug. 2024), pp. 1–22.

Contents

I can still reason—I studied mathematics, which is the madness of reason—[...]

— Clarice Lispector

1

Introduction

A sequence of random processes is *exchangeable* if it is in some sense invariant under permutation. The interpretation of *random process* here is very broad, and the notion of *invariant under permutation* varies depending on the context. One might consider exchangeable events, random variables, random matrices, measures, or quantum states. If $(E_i)_{n\in\mathbb{N}}$ is a sequence with some kind of probabilistic behaviour, then it is exchangeable if and only if, for any $n \in \mathbb{N}$, and any permutation, $\sigma \in \mathcal{S}_n$, in the symmetric group of $\{1, \ldots, n\}$, the likelihood of a given outcome of the sequence $E_1, \ldots, E_n$ is the same as that of $E_{\sigma(1)}, \ldots, E_{\sigma(n)}$. In quantum contexts, at least in this work, this will be equality of states; in the context of classical probability, this translates to equality in distribution.

De Finetti theorems provide representations for such exchangeable structures, and in this thesis we will go about showing they do so in a way that is canonical. This will be done by presenting a number of theorems in categories of probabilistic and quantum processes classifying these representations of the traditional de Finetti theorems via universal properties up to unique isomorphism. Specifically, we will be proving the existence of (unique) (co)limits, final coalgebras and initial transition algebras.

So what can one say of exchangeable processes? Exchangeability is certainly a strong constraint. Processes must have identical outcome spaces. Each individual

E_i must be indistinguishable from the others (To see that E_1 and E_i are identical in outcome, take a permutation for which $\sigma(1) = i$ and marginalise over the remaining E_js).

Under each interpretation of random process, there is some notion of repeated, identical process without interference between instances: independent events; independent and identically distributed (i.i.d.) random variables; the iterated product of a measure with itself; the iterated product of a quantum state. They all exhibit exchangeability. One may be forgiven for suggesting that *all* exchangeable processes might be as such.

They are not. There are non-trivial exchangeable processes in all the above contexts. One way of immediately generating a lot of exchangeable processes is by mixing i.i.d. ones. Suppose Gilda has a bag C of irregular coins. From this bag, she takes a coin and then proceeds to flip it forever. Let us suppose that each coin $c \in C$ has a predetermined bias towards heads of p_c. Then the likelihood that she flips heads on her first flip is

$$\sum_{c \in C} \frac{p_c}{|C|} = \frac{1}{|C|} \sum_{c \in C} p_c \tag{1.1}$$

since she chooses coin c with probability $\frac{1}{|C|}$. Similarly, the chance that she flips a given sequence of heads and tails of length n with h occurrences of heads is

$$\frac{1}{|C|} \sum_{c \in C} p_c^h \left(1 - p_c\right)^{(n-h)} \tag{1.2}$$

since she chooses coin c with probability $\frac{1}{|C|}$ and then flips the appropriate sequence with probability $p_c^h \left(1 - p_c\right)^{(n-h)}$.

This process, then, is exchangeable, and it is certainly not i.i.d. since over the course of the flipping something can be inferred about the coin that has been drawn. If the first ninety-two flips turn up heads, Gilda has better information than she did at the beginning to predict the outcome of the ninety-third[1].

There is clearly an echo of an i.i.d. sequence here, though. Indeed, there is something *conditionally* i.i.d. about the sequence: if she knew which coin $c \in C$

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[1] or to doubt the nature of reality.

she was flipping, and its bias p_c, Gilda could model the flips as a sequence of i.i.d. Bernoulli random variables each with probability of heads p_c.

Luckily, this information is salvageable from the data of the flips. Via the law of large numbers, the relative frequency of flips of heads tends towards p_c in the limit. If we define the sequence of random variables $(X_1, X_2, \dots)$ with $X_i = \{\text{The } i\text{th flip is heads}\}$ and define $P = \lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n}$, then, given $P = p$, the sequence $(X_1, X_2, \dots)$ is going to be i.i.d. distributed with $X_i \sim \mathrm{Ber}(p)$. Given that she has drawn the coin c, then $P = p_c$ almost surely, but without any information about the coin, $P = p_c$ with probability $\frac{1}{|C|}$. It is not a coincidence that via P we regain this distribution on coins, the uniform distribution on C, that was implicit in our original description of Gilda's procedure.

Another canonical example of an exchangeable process is *Pólya's urn*, or the Pólya-Eggenberger urn model.

Take an urn containing k_b black balls and k_w white balls. Construct a sequence $\{0,1\}^{\mathbb{N}}$ as follows: draw a ball and record a 1 if it is black, a 0 if it is white. Return the ball, and add another of the same colour (for example, if a black ball was drawn, there will now be $k_b + 1$ black balls in the urn). Repeat.

We may define random variables $A_i \in \{0,1\}$ with A_i being the i^{th} element of the sequence.

This sequence of outcomes from Pólya's urn are certainly not independent. The likelihood of outcome A_{n+1} is dependent on those that proceed it: many black balls drawn means many more in the bag and so $A_{n+1} = 1$ is more likely. This models systems where the outcomes are self-reinforcing; the original paper from Pólya and Eggenberger introduced the urn model to model contagion [41]. Given this reinforcement behaviour over time, it is perhaps surprising that Pólya's urn exhibits exchangeability.

A quick calculation shows that a sequence of n draws containing l black balls has the probability

$$\frac{k_b \cdot (k_b+1)\cdots(k_b+l-1)\, k_w(k_w+1)\cdots(k_w+n-l-1)}{n!} \tag{1.3}$$

independent of the order in which the balls are drawn. Thus, no matter the original values of k_b and k_w, the sequence of random variables $(A_1, \dots, A_n)$ is equal in distribution to $\left(A_{\sigma(1)}, \dots, A_{\sigma(n)}\right)$ for any permutation $\sigma \in \mathcal{S}_n$.

Let us take inspiration from the bag-of-coins example above, and consider the random variable $Q = \lim_{n\to\infty} \frac{\sum A_i}{n}$. The right-hand side converges almost surely, and it can be shown that $Q \sim \mathrm{Beta}(k_b, k_w)$ as a random variable. Further, given $Q = q$, the sequence $(A_1, A_2, \dots)$ is once again i.i.d. distributed, with $A_i \sim \mathrm{Ber}(q)$. (For a comprehensive discussion of Pólya's urn and its generalisations, see Mahmoud [117].)

The theorem that explains this behaviour is named after the Italian mathematician Bruno de Finetti and in its original form it says that all exchangeable sequences of real random variables have the shape explored above: for every such sequence $(Y_1, Y_2, \dots)$, there is some random variable R such that, given $R = r$, the Y_is are independent and identically distributed $\mathrm{Ber}(r)$. We might say that the sequence $(Y_1, Y_2, \dots)$ is *mixed i.i.d.* or *conditionally i.i.d.*.

This is not limited to real random variables. All the random processes mentioned at the start of this introduction above have similar classifications. De Finetti himself began with exchangeable events before discussing exchangeable random variables. Exchangeable complex random variables and random matrices follow from real random variables. The case of exchangeable measures and exchangeable quantum states are more interesting extensions.

Let X be a measurable space. Suppose for each $n \in \mathbb{N}$, μ_n is a measure on X^n, constructed in such a way that μ_n is the marginalisation of μ_{n+1} over the last copy of X in the product. Such a sequence is *exchangeable* if μ_n is invariant under the transformations X^σ permuting the factors of the product X^n for permutations σ in the symmetric group $\mathcal{S}_n$. With some restrictions on X and μ_n, and some care about what it means to give a measure on measures, we get a similar classification theorem for these processes: there exists Φ, a probability distribution on measures on X (e.g. our bag of coins above is a distribution on biases), such that μ_n is 'the same' as picking a measure ν at random with Φ and then taking the n times product of ν with itself, ν^n.

Now let $\mathcal{H}$ be a Hilbert space representing a quantum system. Let ρ_n be a sequence of quantum states on the systems $\mathcal{H}^{\otimes n}$, the n-times Hilbert tensor product of $\mathcal{H}$ with itself, constructed in such a way that ρ_n is equal to ρ_{n+1} composed with the partial trace over the last factor of the tensor product. Such a sequence is *exchangeable* if ρ_n is invariant under the transformations of the tensor power $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\otimes \sigma}$, permuting the factors by a permutation $\sigma \in \mathcal{S}_n$. With meaningful definitions of such a sequence and care given to what it means to give a measure on states, there is a classification theorem here too. The result is a classical mixture of quantum product states: there exists μ, a probability distribution on states on $\mathcal{H}$, such that ρ_n is 'the same' as picking a state ω at random with μ and then producing the n times product of ω with itself, $\omega^{\otimes n}$.

These de Finetti theorems were studied extensively throughout the mid-twentieth century. De Finetti wrote an overview of his own work on the subject [34], which will be discussed in section 3.1. Kallenberg provided an extensive account of known results about exchangeable classical processes in 2005 [102] whilst much other work has been done and continues to be done [A sample: 1, 3, 4, 8, 10, 12, 13, 44, 45, 74, 75, 83, 130, 132, 135, 137, 158].

Størmer, Hulanicki and Phelps were the originators of the quantum de Finetti theorem [80, 148]. Since then much work has been done in the areas of quantum, non-commutative and free probability exchangeability [e.g. 19, 23, 30, 55, 78, 79, 95, 110, 111]. Extensions have also been studied via other approaches, like test spaces [11] and causality [64], whilst similar discussions of exchangeability for arrays and graphs have motivated the Aldous-Hoover representation theorem for exchangeable arrays [5, 43, 77, 101] and work on exchangeable graph sequences and graphons [e.g. 22, 36, 37, 93, 94, 98, 116].

Most recently, there has been a number of papers considering exchangeability in the context of categorical probability theory. This is where the work of this thesis is situated.

Contribution of this Work

Categorical probability theory uses the tools of category theory to study probabilistic processes. It is of particular value within theoretical computer science as an avenue for understanding the semantics of probabilistic programming languages.

A natural approach would be to study the category **Meas** of measurable spaces and measurable maps, but it is quickly revealed that **Meas** is not a particularly well-behaved category (for example, it cannot be made Cartesian closed [7]). These days, the field is broad but much of the work that has interested this author is about probability monads and their Kleisli categories (overviewed in section 2.5). Similar to this is the study of Markov categories; monoidal categories with the capacity for copying and discarding information. Probability is encoded in those morphisms for which repeating a process twice is not the same as doing it once and copying the output.

What would it look like to study exchangeability through this lens?

In one direction, work on Kolmogorov products in Markov categories and the development of involutive Markov categories has allowed string-diagrammatic formulations of both classical and quantum de Finetti theorems [48, 50, 53]. In the other, Jacobs and Staton noticed in 2020 that the construction of a measure over measures in de Finetti's theorem resembled composition within a category of probabilistic processes. They proved that in the Kleisli category of the Giry monad, de Finetti's theorem applied to $\{0,1\}$-valued measures gave rise to a diagram for which $\mathcal{G}(\{0,1\}) \cong [0,1]$ was the limit, using multisets to encode exchangeability of measure [91]. This result has inspired more treatments of exchangeability from the categorical perspective [14, 59, 144]. The work of this thesis outlines the author's contribution to this project showing similar limit constructions are possible for the quantum de Finetti theorems of Hulanicki-Phelps and Størmer, and the classical de Finetti theorem of Hewitt and Savage. Multisets are also shown to be a valid way of encoding exchangeability in the classical setting, and restriction to algebras of symmetric tensors as a non-commutative dual to multisets in the quantum setting. The final chapter adapts these results towards another universal construction: that of final coalgebras in the classical setting and initial transition algebras in quantum settings.

Chapter 2 introduces the preliminaries for this work. As with the rest of the thesis, it is written to be readable to a graduate student with a moderate undergraduate knowledge of category theory, measure theory, functional analysis and topology. Notably, no prior knowledge of quantum theory is expected. The largest section of this chapter is section 2.5, where a review of probability monads is provided in section 2.5.1, and the Radon probability monad is introduced in section 2.5.2.

Chapter 3 treats the quantum de Finetti theorems, and lays out the form that these categorified exchangeability theorems take.

- This chapter follows closely the structure of our paper, *Quantum de Finetti Theorems As Categorical Limits, and Limits of State Spaces of C*-Algebras*, co-authored with my supervisor, Prof. Sam Staton, presented as a plenary talk at Quantum Physics and Logic 2022, and published in *EPTCS*[2]. The categorical Størmer quantum de Finetti theorem was first published in this paper.

Much of this chapter is focused on laying out various categories of C*-algebras and their properties. These categories are exactly the right place to study the overlapping classical and quantum probability needed to describe the distributions on state spaces that are the universal object of quantum de Finetti theorems.

The progression from a traditional quantum de Finetti theorem to a categorical one comes from noticing that an exchangeable sequence of states ρ_n on $\mathcal{A}^{\otimes n}$ for a C*-algebra $\mathcal{A}$, under either the maximal or minimal C*-tensor norms, can be described as morphisms $\rho_n \colon \mathcal{A}^{\otimes n} \to \mathbb{C}$ with the following compatibility requirements: they are invariant under the braidings $\mathcal{A}^{\otimes \sigma} \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$ in the sense that $\rho_n = \rho_n \circ \mathcal{A}^{\otimes \sigma}$ and if we write $\iota_{n(n+1)} : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes (n+1)}$ for the maps appending the unit of $\mathcal{A}$ to a tensor (these are dual to the partial traces $\mathcal{H}^{\otimes (n+1)} \to \mathcal{H}^{\otimes n}$), then $\rho_n = \rho_{n+1} \circ \iota_{n(n+1)}$.

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[2] Sam Staton and Ned Summers. 'Quantum de Finetti Theorems as Categorical Limits, and Limits of State Spaces of C*-Algebras'. In: *Electronic Proceedings in Theoretical Computer Science* 394 (Nov. 2023), pp. 400–414.

A *parametrised* exchangeable sequence replaces $\mathbb{C}$ with another C*-algebra $\mathcal{B}$ and asks for completely positive maps $\phi_n \colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ satisfying the same equations: $\phi_n = \phi_n \circ \mathcal{A}^{\otimes \sigma}$ for all n and $\sigma \in \mathcal{S}_n$, and $\phi_{n+1} = \phi_n \circ \iota_{n(n+1)}$. This describes a cocone of a diagram, and so a meaningful question is to ask whether there is a universal such cocone. There is: the C*-algebra of continuous functions from the (compact Hausdorff) state-space of $\mathcal{A}$ to $\mathbb{C}$, $CS(\mathcal{A})$, is the colimit. This, the main result of the chapter, is theorem 3.72.

The proof of this theorem exploits the relationship between the functor that takes a C*-algebra $\mathcal{A}$ to the convex, topological space $S(\mathcal{A})$ of states on it; the functor which takes a compact Hausdorff space X to the commutative C*-algebra of continuous functions $X \to \mathbb{C}$, $C(X)$; and the Radon monad $\mathcal{R}$ on the category of compact Hausdorff spaces **CH**. In particular, the category of Eilenberg-Moore algebras of $\mathcal{R}$ is equivalent to the category of compact Hausdorff topological spaces with a cancellative convex structure, **ConvCH**, and S factors through this category.

The colimit above is constructed as a limit in this category of convex spaces. It uses the facts that **ConvCH** is monadic over **CH**, which is in turn monadic over **Set**, and that, after forgetting the convex space structure, $S(\mathcal{A})$ is just the hom-functor $\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A}, \mathbb{C})$ on an appropriate category of C*-algebras, to construct the de Finetti limits point-by-point using the traditional de Finetti theorems.

Other important steps in this proof are theorem 3.68, that the state-space functor preserves and reflects limits; and theorem 3.71, the categorical quantum Kolmogorov extension theorem, particularly its validity in the category of C*-algebras and completely positive maps.

With these results established, the appropriate tools are now available to consider the classical de Finetti theorem in the style of Jacobs and Staton. This is the first topic of chapter 4.

In a similar manner to in the quantum case, the transition from a traditional de Finetti theorem to a categorical one follows from noticing that an exchangeable sequence of measures μ_n on X^n is the same as a sequence of ‘probabilistic morphisms’ $\{*\} \rightsquigarrow X^n$ which are compatible under the projection maps $X^{n+1} \to X^n$ (which is to say, μ_n is equal to μ_{n+1} projected onto X^n) and invariant under pushforward by the braiding maps $X^\sigma \colon X^n \to X^n$. A *parametrised* exchangeable sequence of measures replaces $\{*\}$ with a space Y, asking for probabilistic maps $Y \rightsquigarrow X^n$ that are also invariant under the braidings, and consistent under projection. Classifying parameterised versions of Pólya's urn is an active area of research [13, 137]. In this thesis, the probability in these maps is introduced by using the Kleisli category of the Radon probability monad, $\mathcal{R}\colon \mathbf{CH} \to \mathbf{CH}$. This chapter states and proves two de Finetti theorems, theorems 4.3 and 4.16, classifying $\mathcal{R}(X)$, the space of Radon measures on a compact Hausdorff space X, as a limit of a diagram representing exchangeable parameterised sequences of measures, the latter being in the language of multisets. This is used as a jumping off point to consider a non-commutative dual to multisets, and as such an additional categorical quantum de Finetti theorem, theorem 4.22.

In section 4.2, de Finetti limits in the Kleisli category of $\mathcal{R}$ are explored. Again the relationship between the Radon monad and categories of C*-algebras is exploited, and by specialising the categorical quantum de Finetti theorem to commutative C*-algebras, which are dual to compact Hausdorff spaces, we get theorem 4.4, a form of de Finetti theorem as a limit in **ConvCH**. This then proves the main theorem of the section, theorem 4.3, a general de Finetti limit theorem for exchangeable sequences of measures on any compact Hausdorff space in $\mathcal{K}l(\mathcal{R})$, canonically constructing $\mathcal{R}(X)$ as this limit. These results are found by instantiating the categorical quantum de Finetti theorems with commutative C*-algebras, dual to compact Hausdorff spaces, and following through the various correspondence between states and measures on these spaces, and the various inclusions and equivalences between the categories $\mathbf{CSt}_{\mathrm{CPU}}, \mathbf{CSt}_{\mathrm{PU}}, \mathcal{K}l(\mathcal{R}), \mathbf{ConvCH}$ and $\mathcal{E}m(\mathcal{R})$.

Additionally important in this section is the statement of the categorical Kolmogorov extension theorem, theorem 4.2.

Section 4.3 reframes these results in terms of multisets. Multisets are unordered lists of elements of a space X, with multiple copies of an element allowed, and as such measures on multisets immediately encode exchangeability. This eliminates the necessity of the braiding maps X^σ in the limits above, and the projections $X^{n+1} \to X^n$ may be replaced with probabilistic draw-and-delete map which removes an element from a multiset at random. This is the diagram used in the work of Jacobs and Staton in **Meas**.

To do the same here, the definitions of multisets as a coequaliser in **Set** are extended to a functor on **CH**. The draw-and-delete maps are shown to exist in $\mathcal{K}l(\mathcal{R})$ and in proposition 4.15 it is shown that spaces of multisets are probabilistic coequalisers of the braiding maps X^σ. The main result is theorem 4.16, a general categorical de Finetti limit for a space X in $\mathcal{K}l(\mathcal{R})$ using a draw-and-delete multiset diagram, again constructing $\mathcal{R}(X)$ as the limit. Section 4.3.4 has a discussion of the similarity of this result to that of Jacobs and Staton.

Finally, in section 4.4, the dual construction is considered in categories of C*-algebras. In this context, spaces of multisets become subsets of symmetric tensors. The main theorem in this section is theorem 4.22, showing a 'quantum multiset' form of the quantum de Finetti limit theorem. Many of the arguments in this section require less complicated mathematical machinery than the corresponding ones for the classical case, but the resulting theorem is a direct generalisation of theorem 4.16.

Chapter 5 approaches the classification of exchangeable structures from the perspective of coalgebras and transition algebras of functors. Both classical and quantum cases are treated. Note, these are not the coalgebras of linear algebra.

In the classical context, using the Kleisli category of the Radon monad, $\mathcal{K}l(\mathcal{R})$, the relevant coalgebras are of the functor $X \times -$. Such a coalgebra is a probabilistic map $Y \rightsquigarrow X \times Y$ which takes a parameter $y \in Y$ and from it produces distributions on an output in X and an updated value of the parameter in Y. By continued application of the coalgebra, lists of outcomes in X can be made from an initial seed parameter, and in this way this coalgebraic approach gives rise to measures on the products X^n. Jacobs and Staton introduced a notion of exchangeability of such a coalgebra and showed that in the case that $X = \{0, 1\}$, $\mathcal{G}(\{0, 1\})$ was the carrier of the final exchangeable coalgebra of the functor in $\mathcal{K}l(\mathcal{G})$. The coalgebra takes a measure in $\mathcal{G}(\{0,1\})$, samples from it to get a result in $\{0,1\}$ and then returns the measure unchanged.

The main result of this chapter is theorem 5.19, which states the same thing but now in $\mathcal{K}l(\mathcal{R})$ and for a general space $X \in \mathbf{CH}$. Perhaps more interesting though is the manner of getting there. This chapter is almost exclusively investigated in the general context of an affine, commutative monad $\mathcal{T}$ on a Cartesian category $\mathbf{C}$. Indeed, the whole chapter is true of any such monad that also admits limit forms of Kolmogorov extension and de Finetti theorems and obeys a simple monomorphism-preserving property. Theorem 5.21 sets out the result in maximal generality.

Section 5.2 sets up the functor $A \times -$ on $\mathcal{K}l(\mathcal{T})$ for an object $A \in \mathbf{C}$ and considers coalgebras $B \rightsquigarrow A \times B$ in $\mathcal{K}l(\mathcal{T})$. A main result of this section is that in the case that $\mathcal{K}l(\mathcal{T})$ has a Kolmogorov extension theorem-like limit, there is a final such coalgebra: *the stream coalgebra*, definition 5.6 and theorem 5.7.

Exchangeable coalgebras are introduced, and it is shown that give rise to exchangeable sequences in the sense of de Finetti in theorem 5.9.

Also definable in this setting is *the sample coalgebra*. Using the strength of the monad $\mathcal{T}$ and the diagonal morphism $(\mathrm{id}, \mathrm{id})\colon \mathcal{T}(A) \to \mathcal{T}(A) \times \mathcal{T}(A)$ of $\mathbf{C}$, the sample coalgebra is the coalgebra $\mathcal{T}(A) \to \mathcal{T}(A) \times \mathcal{T}(A) \to \mathcal{T}(A \times \mathcal{T}(A))$. In the case that $\mathcal{T} = \mathcal{G}$ and $A = \{0,1\}$, this is the final exchangeable coalgebra above. A crucial result of this section, theorem 5.11, is proving a closed form for the iterants of this coalgebra, and the distributions it generates on A^n, from which it is shown that the sample coalgebra is exchangeable, theorem 5.12.

These ideas are specialised to the Kleisli category of the Radon monad, where the main result of theorem 5.19 is then shown. There follows a discussion in section 5.4.1 about the similarity of this result to that of Jacobs and Staton, and a proof that this extends their result, theorem 5.20. The maximal generality result, a fully categorical description of when such a final coalgebra can be constructed, is theorem 5.21.

Finally, in section 5.5 the result is resituated to the quantum setting. The dual objects to exchangeable coalgebras of $X \times -$ in $\mathcal{K}l(\mathcal{R})$, exchangeable transition algebras of the functor $\mathcal{A} \hat{\otimes} -$ in the category of C*-algebras and completely positive maps, are studied. Interestingly, though the results are dual to those of the classical case, there is no monad here to utilise in defining the dual object to the sample coalgebra. As such, the transition algebra of note, *the evaluation transition algebra*, is defined explicitly and algebraically. The main result of the chapter, and the final result of the thesis, is theorem 5.32, a generalisation of theorem 5.19, showing this transition algebra is indeed the initial exchangeable transition algebra of the functor $\mathcal{A} \mathbin{\hat{\otimes}} -$ in $\mathbf{CSt}_{\mathrm{CPU}}$.

By the end of this report we will have shown the existence of two colimits (three, if the auxiliary result in $\mathbf{CSt}_{\mathrm{PU}}$ is counted) and one initial transition algebra in categories of quantum processes, and two limits (three, if the auxiliary result in **ConvCH** is counted) and one final coalgebra in categories of probabilistic processes, classifying exchangeable structures, exhibiting category theory as an important tool in the study of exchangeability theorems.

2
Preliminaries

Contents

This chapter outlines the notation and definitions for what follows. Undergraduate familiarity with measure theory, for example the definition of measures, product σ-algebras and simple functions, is assumed, as is familiarity with basic definitions of topology and abstract algebra. A first course in category theory is also assumed. When they are not cited, category theory definitions are drawn from Riehl [131], which would also provide more than sufficient background for knowledge assumed.

2.1 Common Categories

For a category $\mathbf{C}$, we denote the class of its objects as $\mathrm{ob}(\mathbf{C})$ and the class of morphisms from C to D as $\mathbf{C}\left(C, D\right)$. $\mathbf{C}\left(-, D\right) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ denotes the appropriate hom-functor when $\mathbf{C}$ is locally small. In general, if there is an obvious forgetful functor out of $\mathbf{C}$, it will be denoted by $U_{\mathbf{C}}$.

The category of sets and functions will be denoted by **Set**. The full subcategory of finite sets is **FinSet**. $\mathbf{Vect}_k$ denotes the category of vector spaces over a field k.

A measurable space is a pair (X, Σ_X) of a set X and a σ-algebra of subsets of X, Σ_X. *A measurable map* between measurable spaces $(X, \Sigma_X) \to (Y, \Sigma_Y)$ is a function $f\colon X \to Y$ such that $f^{-1}(B) \in \Sigma_X$ for all $B \in \Sigma_Y$. **Meas** is the category of measurable spaces and measurable maps.

The following two categories, **Inj** and $(\mathbb{N}, \leq)$, will play important parts as index categories for limit theorems. In both, for convenience, we will suppose $0 \notin \mathbb{N}$. If necessary, in this work we will write $\mathbb{N}_0$ for the natural numbers including 0. This is not particularly important and assuming otherwise doesn't change anything about any of the results, as 0 may be used to index a trivial object in every diagram, either the initial object in classical circumstances or the final object in the quantum circumstances.

We denote by $(\mathbb{N}, \leq)$ the poset $\mathbb{N}$ considered as a category. In particular, in this category there exists exactly one morphism $m \to n$ if and only if $m \leq n$. Unless $n = m$, there are no morphisms $n \to m$ in this case.

Definition 2.1 (Category of Finite Ordinals and Injections)**.** The category of finite ordinals and injections*,* **Inj***, has* $\mathrm{ob}(\mathbf{Inj}) = \mathbb{N}$ *and the arrows* $\mathbf{Inj}(m, n)$ *are exactly the injective functions* $\phi\colon \{1, \dots, m\} \to \{1, \dots, n\}$*. These morphisms need not preserve order. Composition is function composition.*

This category is equivalent to the wide subcategory of monomorphisms in **FinSet**.

Every morphism in **Inj** can be decomposed into a composition of the identity injection $\{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ and a (likely non-unique) permutation $\sigma \in \mathcal{S}_n$ on n elements in $\mathbf{Inj}(n, n)$.

Another way of saying this is that these two classes of morphism form a *weak factorisation system* on **Inj**. There is also a *strict factorisation system* on **Inj** where the left morphisms are all isomorphisms (that is, all the permutations on $\{1, \ldots, m\}$) and the right morphisms are order-preserving functions, which is to say every morphism in **Inj** admits a unique factorisation into a permutation followed by an order-preserving injection.

2.2 Category Theory

Definition 2.2 (Categorical Limit)**.** *Let* $\mathcal{D}\colon \mathcal{I} \to \mathbf{C}$ *be a functor. In this context,* $\mathcal{D}$ *is called a* diagram *of shape* $\mathcal{I}$*.* A cone over $\mathcal{D}$ *with* apex $D \in \mathbf{C}$ *is a collection of morphisms,* $\{\phi_i\colon D \to \mathcal{D}(i)\}$ *for each* $i \in \mathcal{I}$ *such that for every morphism* $f\colon i \to j$ *in* $\mathcal{I}$*,* $\phi_j = \mathcal{D}(f) \circ \phi_i$*.*

A limit *of the diagram* $\mathcal{D}$*, if one exists, is a cone over* $\mathcal{D}$ *with apex* $\lim \mathcal{D}$*,* $\{\lambda_i\colon \lim \mathcal{D} \to \mathcal{D}(i)\}$*, such that if* $\{\phi_i\colon D \to \mathcal{D}(i)\}$ *is another cone of* $\mathcal{D}$*, there exists a unique* $\lambda\colon \lim \mathcal{D} \to D$ *such that for all* $i \in \mathcal{I}$*,* $\lambda_i = \phi_i \circ \lambda$*. Any limit of* $\mathcal{D}$ *is unique up to unique isomorphism, so we will often write* the *limit of the diagram* $\mathcal{D}$*.*

A limit in the opposite category is a colimit*, and is denoted* $\operatorname{colim} \mathcal{D}$*.*

Definition 2.3 (Preserved, Reflected and Created Limit)**.** *A functor* $F\colon \mathbf{C} \to \mathbf{D}$ *is said to* preserve limits *if for any diagram* $\mathcal{D}\colon \mathcal{I} \to \mathbf{C}$ *with a limit* $\lim \mathcal{D}$*, the image of the limit under* F *is the limit to the* $\mathcal{I}$*-shaped diagram* $F \circ \mathcal{D}\colon \mathcal{I} \to \mathbf{D}$*. In other words,* $F(\lim \mathcal{D}) = \lim (F \circ \mathcal{D})$*.*

F *is said to* reflect limits *if a cone over* $\mathcal{D}$ *in* $\mathbf{C}$ *whose image under* F *in* $\mathbf{D}$ *is a limit for* $F \circ \mathcal{D}$*, is a limit of* $\mathcal{D}$ *itself.*

F *is said to* create limits *if whenever* $F \circ \mathcal{D}$ *has a limit in* $\mathbf{D}$*, there exists a cone in* $\mathbf{C}$ *of* $\mathcal{D}$ *for which this limit is the image under* F*, and this limit is reflected.*

Proposition 2.4 ([131, Lma. 3.3.5.])**.** *Full and faithful functors reflect all limits and colimits in their codomain.*

Proposition 2.5 ([131, Thm. 3.4.11])**.** *For a locally small category* $\mathbf{C}$ *and an object* $D \in \mathbf{C}$*, the hom-functor* $\mathbf{C}(-, D) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *preserves limits (which is to say, it carries colimits in* $\mathbf{C}$ *to limits in* $\mathbf{Set}$*).*

Definition 2.6 (Monad)**.** *A* monad *on a category* $\mathbf{C}$ *consists of an endofunctor* $\mathcal{T}\colon \mathbf{C} \to \mathbf{C}$ *with*

1. *A natural transformation* $\eta\colon \mathbf{1_C} \Rightarrow \mathcal{T}$ *called* the unit of the monad*.*
2. *A natural transformation* $\mu\colon \mathcal{T}^2 \Rightarrow \mathcal{T}$ *called* the multiplication of the monad*.*

The following diagrams must commute:

$$\begin{tikzcd}
\mathcal{T}^3 \arrow[r, Rightarrow, "\mathcal{T}\mu"] \arrow[d, Rightarrow, "\mu\mathcal{T}"'] & \mathcal{T}^2 \arrow[d, Rightarrow, "\mu"] & & \mathcal{T} \arrow[r, Rightarrow, "\eta\mathcal{T}"] \arrow[dr, Rightarrow, "\mathbf{1}_{\mathcal{T}}"'] & \mathcal{T}^2 \arrow[d, Rightarrow, "\eta"] & \mathcal{T} \arrow[l, Rightarrow, "\mathcal{T}\eta"'] \arrow[dl, Rightarrow, "\mathbf{1}_{\mathcal{T}}"] \\
\mathcal{T}^2 \arrow[r, Rightarrow, "\mu"'] & \mathcal{T} & & & \mathcal{T} &
\end{tikzcd} \tag{2.1}$$

Definition 2.7 (Kleisli Category)**.** *Given a monad* $\mathcal{T}\colon \mathbf{C} \to \mathbf{C}$*, the* Kleisli category of $\mathcal{T}$ *is the category* $\mathcal{K}l(\mathcal{T})$ *with* $\mathrm{ob}(\mathcal{K}l(\mathcal{T})) = \mathrm{ob}(\mathbf{C})$*. A morphism from* C *to* D*, denoted by* $C \rightsquigarrow D$*, is a morphism* $C \to \mathcal{T}(D)$ *in* $\mathbf{C}$*.*

On an object $C \in \mathrm{ob}(\mathbf{C})$*, the identity in* $\mathcal{K}l(\mathcal{T})$ *is the unit of the monad:* $\eta\colon C \rightsquigarrow C$*.*

Composition of two morphisms $f\colon C \rightsquigarrow D$ *and* $g\colon D \rightsquigarrow E$ *is given by the following composition in* $\mathbf{C}$*:*

$$c \xrightarrow{f} \mathcal{T}(D) \xrightarrow{\mathcal{T}(g)} \mathcal{T}^2(E) \xrightarrow{\mu_E} \mathcal{T}(E) \tag{2.2}$$

Definition 2.8 (Eilenberg-Moore Category)**.** An Eilenberg-Moore algebra of a monad $\mathcal{T}\colon \mathbf{C} \to \mathbf{C}$*, or* a $\mathcal{T}$-algebra*, is a morphism* $a\colon \mathcal{T}(C) \to C$ *in* $\mathbf{C}$ *such that*

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & \mathcal{T}(C) \arrow[d, "a"] & & \mathcal{T}^2(C) \arrow[r, "\mu_C"] \arrow[d, "\mathcal{T}(a)"'] & \mathcal{T}(C) \arrow[d, "a"] \\
& C & & \mathcal{T}(C) \arrow[r, "a"'] & C
\end{tikzcd} \tag{2.3}$$

commute. The object $C \in \mathbf{C}$ *is called the* carrier of the algebra a*.*

A morphism of $\mathcal{T}$-algebras *from* $a\colon \mathcal{T}(C) \to C$ *to* $b\colon \mathcal{T}(D) \to D$ *is a morphism* $f\colon C \to D$ *in* $\mathbf{C}$ *such that the diagram*

$$\begin{tikzcd}
\mathcal{T}(C) \arrow[r, "\mathcal{T}(f)"] \arrow[d, "a"'] & \mathcal{T}(D) \arrow[d, "b"] \\
C \arrow[r, "f"'] & D
\end{tikzcd} \tag{2.4}$$

Given a monad $\mathcal{T}\colon \mathbf{C} \to \mathbf{C}$*, the* Eilenberg-Moore category of $\mathcal{T}$ *is the category of Eilenberg-Moore algebras and their morphisms. It is denoted by* $\mathcal{E}m(\mathcal{T})$*.*

Given an object $C \in \mathbf{C}$, *the free* $\mathcal{T}$*-algebra associated with* C is the algebra $\mu_C\colon \mathcal{T}^2(C) \to \mathcal{T}(C)$. The Kleisli category $\mathcal{K}l(\mathcal{T})$ embeds as a full subcategory of $\mathcal{E}m(\mathcal{T})$ via the mapping of $C \in \mathbf{C}$ to the free $\mathcal{T}$-algebra μ_C. A morphism $f\colon C \rightsquigarrow D$ is taken to the morphism of free algebras $\mathcal{T}(f)\colon \mathcal{T}(C) \to \mathcal{T}(D)$.

Definition 2.9 (Monadic Functor)**.** *A functor* $F\colon C \to D$ *is called* monadic *if it has a left adjoint* $U\colon D \to C$ *that induces an equivalence* $D \cong \mathcal{E}m(UF)$*.*

Theorem 2.10 ([131, Thm. 5.6.5])**.** *A monadic functor* $F\colon C \to D$ *creates all limits that* D *has.*

Definition 2.11 (Monoidal Category)**.** *A* monoidal category *is a triple* $(\mathbf{C}, \otimes, \mathbf{I})$ *consisting of a category* $\mathbf{C}$*, a (binary) functor called the* monoidal product $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *(infixed) and a designated* unit object $\mathbf{I} \in \mathbf{C}$*, with specified natural isomorphisms* $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ *and* $\mathbf{I} \otimes A \cong A \otimes \mathbf{I} \cong A$*.*

If there is an additional given natural isomorphism, called the swap map *or the* braiding*,*

$$\mathit{swap}\colon A \otimes B \cong B \otimes A \tag{2.5}$$

with $\mathit{swap}^2 = \mathrm{id}$*, then* $(\mathbf{C}, \otimes, \mathbf{I})$ *is called a* symmetric monoidal category*.*

The natural isomorphisms are required to commute under certain coherence conditions. These enforce that different orderings of the composition of associativity, unit and swap isomorphisms agree when we would expect them to.

Given a symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{I})$*, there exist (via composition of the swap maps on different factors)* braiding maps *which reorder the factors of an arbitrary monoidal product for any permutation* $\sigma \in \mathcal{S}_n$*. In fact, these are isomorphisms, for example*

$$C_1 \otimes C_2 \otimes C_3 \otimes C_4 \xrightarrow{\sim} C_3 \otimes C_4 \otimes C_2 \otimes C_1 \tag{2.6}$$

If all $C_i = C$*, then we denote by* $C^{\otimes\sigma}$ *the braiding map of* $\underbrace{C \otimes \cdots \otimes C}_{n \text{ times}}$ *associated with* σ*.*

Definition 2.12 (Cartesian Symmetric Monoidal Category)**.** *If a category* $\mathbf{C}$ *has all binary products and a terminal object* $\mathbf{1}$*, then* $(\mathbf{C}, \times, \mathbf{1})$ *naturally has the structure of a symmetric monoidal category. Such a category is called* Cartesian symmetric monoidal*, or simply* Cartesian*.*

The existence of a monoidal structure can interact with the structure of a monad. The definitions here are drawn from Perrone [127, Chap. 6.4].

Definition 2.13 (Strong and Commutative Monads)**.** *Let* $\mathcal{T}$ *be a monad on a monoidal category* $(\mathbf{C}, \otimes, \mathbf{I})$*. A* (left) strength for $\mathcal{T}$ *is a natural transformation* $l_{C,D} \colon C \otimes \mathcal{T}(D) \to \mathcal{T}(C \otimes D)$ *satisfying certain compatibility equations with products with the monoidal unit; consecutive applications of the strength and associativity of the monoidal product; the monad unit; and the monad multiplication. A* right strength for $\mathcal{T}$ *is a natural transformation* $r_{C,D} \colon \mathcal{T}(C) \otimes D \to \mathcal{T}(C \otimes D)$ *with analogous equations. A monad equipped with left and right strengths is called a* strong monad*.*

If $\mathcal{T}$ *is a monad on a symmetric monoidal category, then a left strength is sufficient, defining the right strength with use of the swap map.*

$$\begin{tikzcd}
A \otimes \mathcal{T}(B) \arrow[r, "swap", "\sim"'] \arrow[d, "r_{A,B}"'] & \mathcal{T}(B) \otimes A \arrow[d, "l_{B,A}"] \\
\mathcal{T}(A \otimes B) & \mathcal{T}(B \otimes A), \arrow[l, "\sim"', "\mathcal{T}(swap)"]
\end{tikzcd} \tag{2.7}$$

In what follows, a strong monad will always be a monad on a symmetric monoidal category with a left strength.

A strong monad $\mathcal{T}$ *is called* commutative *if the following diagram commutes:*

$$\begin{tikzcd}
& \mathcal{T}(C) \otimes \mathcal{T}(D) \arrow[dl, "r_{\mathcal{T}(C),D}"'] \arrow[dr, "l_{C,\mathcal{T}(D)}"] & \\
\mathcal{T}(\mathcal{T}(C) \otimes D) \arrow[d, "\mathcal{T}(l_{C,D})"'] & & \mathcal{T}(C \otimes \mathcal{T}(D)) \arrow[d, "\mathcal{T}(r_{C,D})"] \\
\mathcal{T}^2(C \otimes D) \arrow[dr, "\mu"'] & & \mathcal{T}^2(C \otimes D) \arrow[dl, "\mu"] \\
& \mathcal{T}(C \otimes D). &
\end{tikzcd} \tag{2.8}$$

If $\mathcal{T}$ is commutative, then the natural transformation $\nabla_{C,D}\colon \mathcal{T}(C)\otimes\mathcal{T}(D)\to\mathcal{T}(C\otimes D)$ defined from the top to the bottom is called the mediator *of $\mathcal{T}$.*

Commutative monads are equivalent to the notion, beyond our scope here, of a monoidal monad [61, Sec. A.4]. This is a valuable note; we will invoke a coherence theorem for monoidal monads in chapter 5.

Definition 2.14 (Affine monad, [107, Thm. 2.1])**.** *A commutative monad $\mathcal{T}$ on a Cartesian category $(\mathbf{C},\times,\mathbf{1})$ is called* affine *if $\mathcal{T}(\mathbf{1})\cong\mathbf{1}$.*

Equivalently, $\mathcal{T}$ is affine if and only if the following diagram commutes for all $C, D \in \mathbf{C}$:

$$\begin{tikzcd}
\mathcal{T}(C)\times\mathcal{T}(D) \arrow[r, "\nabla_{C,D}"] \arrow[dr, "\mathrm{id}"'] & \mathcal{T}(C\times D) \arrow[d, "(\mathcal{T}(\pi_C){,}\mathcal{T}(\pi_D))"] \\
& \mathcal{T}(C)\times\mathcal{T}(D)
\end{tikzcd} \tag{2.9}$$

2.3 Topology

The category of topological spaces and continuous maps is denoted by **Top**. An important subcategory of **Top** is as follows:

Definition 2.15 (Category of Compact Hausdorff Topological Spaces)**.** *Recall that a topological space $(X,\mathcal{O})$ is* compact *if every open cover of X, i.e. a set of open sets $\mathfrak{D}\subset\mathcal{O}$ such that $\bigcup_{U\in\mathfrak{D}} U = X$, has a finite subcover: a finite subset $C\subset\mathfrak{D}$ which also covers X.*

A topological space X is Hausdorff *if points can be separated by open neighbourhoods: given $x, y\in X$, there exists open sets $U_x, U_y\subset X$ such that $x\in U_x$, $y\in U_y$ and $U_x\cap U_y=\varnothing$.*

The category of compact Hausdorff topological spaces, *notated by* **CH***, has compact Hausdorff topological spaces as its objects and continuous maps as its morphisms. It is full subcategory of* **Top**.

CH comes equipped with the forgetful functor $U_{\mathbf{CH}} \colon \mathbf{CH} \to \mathbf{Set}$ which takes a space to its set of points.

Compact Hausdorff spaces are generally very well-behaved spaces. In fact, in many respects they behave more like algebraic objects than like the more-frequently painful objects that occur in the study of other topological spaces. They have, informally, a *good amount of open sets.* Sufficiently many open sets to separate points, but not so many open sets that functions can run away to infinity. This algebraic-ness is summarised with the following result from Manes.

Theorem 2.16 (**CH** is Monadic over **Set**, [118])**.** *The forgetful functor* $U_{\mathbf{CH}}$ *is monadic.*

We will also be interested in compact Hausdorff spaces with particular convex structures.

Definition 2.17. *A subset W of a vector space V is called* convex *if, for all $v, u \in W$ and $\lambda \in [0,1]$, $\lambda v + (1-\lambda)\, u \in W$.*

W is called balanced *if, for all $v \in W$ and scalars $c \in \mathbb{C}$ with $|c| \leq 1$, $c \cdot v \in W$.*

A topological vector space V (that is, a vector space with an attached topology on vectors such that addition and scalar multiplication are continuous) is called locally convex *if its topology has a basis of convex, balanced sets.*

Definition 2.18 ([58, 150])**.** *The category* **ConvCH** *has as its objects pairs $(\mathbf{W}, V)$ of a locally convex Hausdorff topological vector space V and a compact, convex subset $\mathbf{W} \subset V$. Morphisms $(\mathbf{W}_1, V_1) \to (\mathbf{W}_2, V_2)$ are linear, continuous maps $\mathbf{W}_1 \to \mathbf{W}_2$.*

We will almost always refer to $(\mathbf{W}, V)$ simply by $\mathbf{W}$ and will rarely specify V, since morphisms do not refer to elements of $V \setminus \mathbf{W}$.

One may define *an abstract convex, compact Hausdorff space* as an object $X \in \mathbf{CH}$ equipped with a map taking points $x, y \in X$ and $\lambda \in [0,1]$ to a point $\lambda x + (1-\lambda)\, y \in X$ obeying reasonable conditions. Such a space with convex structure is *cancellative* if $\lambda x + (1-\lambda)\, y = \lambda x + (1-\lambda)\, z$ implies $y = z$ for any points x, y, z and $\lambda \in (0,1)$. The abstract convex, compact Hausdorff spaces realisable as $\mathbf{W}$ for some $(\mathbf{W}, V) \in \mathbf{ConvCH}$ are exactly the cancellative ones [146, Thm. 2]. A map between such spaces that is linear, in the sense that $f(\lambda x + (1-\lambda)\, y) = \lambda f(x) + (1-\lambda)\, f(y)$, is called *affine.*

We will write $U_{\mathbf{ConvCH}} \colon \mathbf{ConvCH} \to \mathbf{CH}$ for the forgetful functor which forgets the convex structure (but remembers the topology).

Definition 2.19 (Extreme Points of a Convex Space)**.** *Given* $(\mathbf{W}, V) \in \mathbf{ConvCH}$, the set of extreme points of $\mathbf{W}$*, denoted* $\partial\mathbf{W}$*, is the set of elements that cannot be decomposed into a convex combination of others. In other words,* $x \in \mathbf{W}$ *is* an extreme point*, or* extremal*, if* $\lambda y + (1-\lambda)\, z = x$ *implies that* $y = z = x$ *for any* $y, z \in \mathbf{W}$*.*

2.4 Multisets

Multisets represent an alternative approach to encoding exchangeability of measures. They may also be thought of bags or urns of elements from a set, with multiplicity of an element allowed. They may be defined in a number of equivalent ways. We follow the notation and style of Jacobs [87, 88, 91, 92], though the history of multisets is much longer.

Definition 2.20 (Multiset)**.** *For a set* A*, a* multiset (of elements of A) *or an* A-multiset *is a finitely-supported map* $\phi \colon A \to \mathbb{N}_0$*, where* $\mathbb{N}_0$ *is the set of natural numbers including* 0*.*

For $B \subset A$ *finite, and a function* $n \colon B \to \mathbb{N}_0$*, We write* $\sum_{a \in B} n(a)\, |a\rangle$ *for the multiset*

$$\begin{aligned} \sum_{a \in B} n(a)\, |a\rangle \, : A \to \mathbb{N}_0 \\ a \mapsto \begin{cases} n(a) & \text{if } a \in B \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.10}$$

Recall to say such a map is finitely supported is to say that $\mathrm{supp}(\phi) := \{a \in A \,|\, \phi(a) \neq 0\}$ is finite.

Definition 2.21. *For a set A and $n \in \mathbb{N}_0$, the* set of A-multisets of order n *is the set*

$$\mathcal{M}_n(A) := \left\{ \phi\colon A \to \mathbb{N}_0 \,\middle|\, \sum_{a \in A} \phi(a) = n \right\}. \tag{2.11}$$

Note that any $\phi \in \mathcal{M}_n(A)$ must have finite support and thus is indeed a multiset. The set of all A-multisets *is*

$$\mathcal{M}(A) := \bigcup_{n \in \mathbb{N}_0} \mathcal{M}_n(A). \tag{2.12}$$

By counting occurrences, but not order, multisets describe exchangeability of sequences of elements of A. Indeed, we may equivalently consider a multiset $\phi \in \mathcal{M}_n(A)$ as an equivalence class of elements of A^n. The equivalence relation is that n-tuples are permutations of each other:

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \iff \exists \sigma \in \mathcal{S}_n \text{ with } x_{\sigma(i)} = y_i \text{ for all } i \in \{1, \ldots, n\} \tag{2.13}$$

and a multiset ϕ is represented by the equivalence class of all those $(x_1, \ldots, x_n)$ with $\phi(a) = \sum_{i=1}^n \{a = x_i\}$. Note, $\{P\}$ is the characteristic function of a proposition, giving 1 if P, and 0 otherwise.

Categorically, this is a statement about coequalisers:

Definition 2.22 (Multisets, alternative definition)**.** *Let A be a set. The set of A-*Multisets of order n*, $\mathcal{M}_n(A)$, is the coequaliser of the braiding maps $A^\sigma\colon A^n \to A^n$ for $\sigma \in \mathcal{S}_n$.*

$$\begin{tikzcd} A^n \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & A^n \arrow[r, "\mathrm{acc}"] & \mathcal{M}_n(A) \end{tikzcd} \tag{2.14}$$

The coequaliser map $\mathrm{acc}\colon A^n \to \mathcal{M}_n(A)$ *is called* accumulation*. It takes an n-tuple $(x_1, \ldots, x_n)$ to the multiset $\phi(a) = \sum_{i=1}^n \{a = x_i\}$ which counts the occurrences of each element in the list and forgets its order.*

As in eq. (2.12), the set of all multisets is the union of $\mathcal{M}_n(A)$ over all $n \in \mathbb{N}_0$.

Definition 2.23 (Multiset Monad on **Set**)**.** *The function $\mathcal{M}$ can be given the structure of a monad, the* multiset monad*.*

As a functor, for a function $f\colon A \to B$

$$\begin{aligned} \mathcal{M}(f)\colon \mathcal{M}(A) &\to \mathcal{M}(B) \\ \sum_{a\in\operatorname{supp}(\phi)} \phi(a)\,|a\rangle &\mapsto \sum_{a\in\operatorname{supp}(\phi)} \phi(a)\,|f(a)\rangle\,, \end{aligned} \tag{2.15}$$

where the right-hand side is considered as a formal sum of $|b\rangle$s, and so like terms are collected.

As a monad, its unit is given by singleton multisets.

$$\begin{aligned} \eta_A\colon A &\to \mathcal{M}(A) \\ a &\mapsto 1\,|a\rangle \end{aligned} \tag{2.16}$$

The multiplication of the monad is just the collation of multisets: given a collection of bags of elements of A, collect all their elements into one bag.

$$\begin{aligned} \mu_{\mathcal{M}}\colon \mathcal{M}^2(A) &\to \mathcal{M}(A) \\ [(\phi_1,\ldots,\phi_n)] &\mapsto \sum_{a\in\bigcup\operatorname{supp}(\phi_i)} \left(\sum_{i=1}^{n}\phi_i(a)\right)|a\rangle \end{aligned} \tag{2.17}$$

where $[\cdot]$ on the left hand represents the equivalence class under the quotient in definition 2.22.

2.5 Radon Probability Monad

2.5.1 A review of probability monads

To call a monad a probability monad is an informal classification, rather than a mathematically rigorous one. The monad takes some object A to an object $\mathcal{P}A$ representing probability measures on that object. The unit of the monad takes a point of A to the probability measure supported solely at that point. The multiplication is some form of averaging over the measures produced: a measure on $\mathcal{P}A$, or a random measure on A, can be collapsed to its expected distribution on A. Sometimes one might consider a more general space of unnormalised, signed, complex-valued or other measures, $\mathcal{M}(A)$.

Example 2.24. *The distribution monad* $\mathcal{D}\colon \mathbf{Set} \to \mathbf{Set}$ *takes a set* A *to the set*

$$\mathcal{D}(A) := \left\{ p\colon A \to [0,1] \,\middle|\, p^{-1}\left((0,1]\right) \text{ is finite, and } \sum_{a\in A} p(a) = 1 \right\}. \tag{2.18}$$

The elements of $\mathcal{D}(A)$ *are just probability mass functions on* A *with finite support. The monad unit takes* $a \in A$ *to* $\eta_A(a)(b) := \delta_a(b) = \{a = b\}$.

Multiplication averages over a random distribution: given $P \in \mathcal{D}^2(A)$ *and* $a \in A$,

$$\mu_{\mathcal{D}}(P)(a) = \sum_{p\in\mathcal{D}(A)} P(p)\, p(a). \tag{2.19}$$

In other words, the likelihood of drawing a *from* $\mu_{\mathcal{D}}(P)$ *is the chance of taking a random* $p \in \mathcal{D}(A)$ *using* P, *and then getting* a *using* p.

There are many monads of this form. The most basic is the distribution monad on **Set** above [e.g. 46, Def. 3.3, 84]. Giry gave her name to two more, one on Polish spaces and the other on **Meas** [60, 115]. Similar is the probability monad on quasi-Borel spaces by Heunen et al. [74], with another sheaf-like approach being explored by Simpson [141]. The Kantorovich probability monad lives in the category of complete metric spaces and short maps, described by Fritz and Perrone [51]. This thesis is concerned with the Radon monad which lives on **CH** and was described by Semadeni and Swirszcz [140, 150], before being explored more recently Furber and Jacobs [58], approaching this (as will be explored) through the relationship with C*-algebras. The Radon monad may also be constructed with respect to ordered spaces, as done by Keimel [104], whilst the operator algebra approach also suggests probability monad constructions on W*-algebras [56]. Also in [58] is explored the expectation monad, which sits somewhere between the distribution monad and the continuation monad (which is not a probability monad) [89]. Similar to probability monads are monads of valuation on topological spaces, and these are of interest in domain theory [e.g. 52, 68, 97, 156]. Most recently, Kristel and Peterseim have proposed the Baire monad, which subsumes the Giry monad on **Meas**, and the Riesz monad subsuming the Baire (thus, Giry) and Radon monads [113]. Interactions with other monads are also studied, such as Dash's probabilistic point process monad or interactions with non-determinism and termination from Mio et al. [32, 33, 121].

There is yet to be stated a satisfying set of categorical criteria to summarise what is shared by these monads, though they mostly arise as affine, commutative monads on Cartesian categories. The canonical exploration of the use of commutative monads to this end is by Kock [108]. Jacobs connects these with commutative effectuses, and focuses on further properties of being strongly affine and partially additive [86]. Relationships with codensity monads have also been explored by Avery and Van Belle [9, 154], and applications to programming language semantics explored directly by, e.g., Jia et al. [96].

A promising area in the study of categorical probability theory are *Markov categories*, named by Fritz in [47]. These are semicartesian monoidal categories (monoidal categories whose monoidal units are terminal) with commutative comonoid structures on objects. A commutative comonoid A is just the categorical opposite to a commutative monoid. In place of multiplication it has *comultiplcation* $A \to A \otimes A$, called *copying*, whilst the counit $A \to \mathbf{I}$ is *discarding*. In such a context, probabilistic behaviour is represented by the failure of these copy maps to be natural: rolling a die and then copying the outcome is not the same as rolling two dice. Affine, commutative monads on Cartesian categories give rise to Markov categories as their Kleisli categories. A *representable* Markov category is a category with such a probability monad which gives rise to the structure [54]. The study of non-commutative/quantum extensions of Markov categories is ongoing, with at least two competing but equivalent notions currently proposed by Parzyngat, and Fritz and Lorenzin respectively [50, 123, 124], and similar approaches have been taken towards these representability conditions. In both the classical and quantum cases, de Finetti theorems, which will play a crucial part in this thesis, may be utilised towards representability, explored in [50], off the back of [48].

2.5.2 The Radon Monad

So we turn to defining the Radon probability monad. This is a monad on the category of compact Hausdorff spaces, **CH**, and relies on the topological properties of the spaces in this category to both build measures and to topologise the space of such measures.

Definition 2.25 (Product and Borel Measures)**.** *Given a measurable space* (X, Σ_X)*,* a measure on (X, Σ_X) *is a countably-additive function* $\mu\colon \Sigma_X \to \mathbb{R}_{\geq 0}$ *for which* $\mu(\varnothing) = 0$*. It is a* probability measure *if* $\mu(X) = 1$*.*

Given two measures μ_i *on* (X_i, Σ_{X_i}) *for* $i = 1, 2$*, a* product measure $\mu_1 \times \mu_2$ *is a measure on* $(X_1 \times X_2, \Sigma_{X_1} \otimes \Sigma_{X_2})$ *with the property that* $\mu_1 \times \mu_2 (A_1 \times A_2) = (\mu_1(A_1))(\mu_2(A_2))$*, where* $\Sigma_{X_1} \otimes \Sigma_{X_2}$ *is the* σ*-algebra generated by sets of the form* $A_1 \times A_2$ *for* $A_i \in \Sigma_{X_i}$*. In the case of probability measures, there is a unique product measure with this property [e.g. 65, P.157, Thm. B].*

Given a topological space X*, the* σ*-algebra generated by its open sets is called the* Borel σ-algebra *and is written* $B(X)$*.*

A Borel measure *is a measure on* $(X, B(X))$*. It is a* Borel probability measure *if the total measure of* X *is* 1*.*

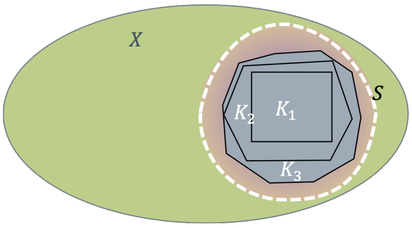
When we study measures on these spaces, we may use Borel measures as usual, but intuitively the compact subsets of a space $X \in \mathbf{CH}$ now play the role of sub-objects and as such it is more useful to consider Borel measures that allow approximation over such compact subsets:

Definition 2.26 (Radon Measure)**.** *Let* $X \in \mathbf{CH}$ *be a compact Hausdorff space. A Borel measure on* X*,* $\mu\colon B(X) \to \mathbb{R}_{\geq 0}$ *is called* Radon *if it is* inner regular*: for all* $S \in B(X)$

$$\mu(S) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \mu(K). \tag{2.20}$$

We are concerned with Radon probability measures*: Radon measures of total mass one.*

Radon measures are exactly those measures that are defined by and play well with the compact subsets of a space.



There are a number of equivalent ways to define the Radon monad. We follow Van Belle [154, Chap. 6], but similar discussions can be found in [86, 104, 140, 150].

Definition 2.27 (Radon Monad)**.** *The underlying functor* $\mathcal{R}\colon \mathbf{CH} \to \mathbf{CH}$ *takes a topological space* X *to the set*

$$\mathcal{R}(X) := \{\mu\colon B(X) \to [0,1] \mid \mu \textit{ is a Radon probability measure}\} \tag{2.21}$$

with the coarsest topology such that the evaluation maps $\mathrm{ev}_\phi\colon \mathcal{R}(X) \to \mathbb{C}$*, which takes a measure* $\mu \in \mathcal{R}(X)$ *to* $\int_X \phi \,\mathrm{d}\mu$ *for each continuous function* $\phi\colon X \to \mathbb{C}$*, are continuous.*

It is non-trivially true that with this topology, $\mathcal{R}(X)$ *is itself a compact Hausdorff space.*

On morphisms, $\mathcal{R}$ *acts by pushforward. For a continuous function* $f\colon X \to Y$*, the corresponding action on measures takes* $\mu \in \mathcal{R}(X)$ *to the pushforward measure* $f_*\mu \in \mathcal{R}(Y)$ *with*

$$\mathcal{R}(f)(\mu)(A) := f_*\mu(A) = \mu\left(f^{-1}(A)\right), \tag{2.22}$$

for $A \in B(Y)$*.*

$\mathcal{R}$ *is a probability monad and its definition follows that of the distribution probability monad (example 2.24), now extended via integration to infinitely supported measures.*

The unit of the monad $\eta_{\mathcal{R}}\colon \mathbf{1} \Rightarrow \mathcal{R}$ *is given by the delta distributions* $\eta_{\mathcal{R}}(X) = \delta^X_-\colon X \to \mathcal{R}(X)$ *with*

$$\delta^X_x(A) = \begin{cases} 1 & \textit{for } x \in A \\ 0 & \textit{otherwise.} \end{cases} \tag{2.23}$$

The multiplication of the monad is again averaging over a random distribution. Given $X \in \mathbf{CH}$ *and a measure* $\Phi \in \mathcal{R}^2(X)$*, we define the corresponding measure* $\mathrm{m}_{\mathcal{R}}(\Phi) \in \mathcal{R}(X)$ *by*

$$\mathrm{m}_{\mathcal{R}}(\Phi)(A) = \int_{\mu \in \mathcal{R}(X)} \mu(A)\,\mathrm{d}\Phi(\mu). \tag{2.24}$$

Both $\eta_{\mathcal{R}}$ *and* $\mathrm{m}_{\mathcal{R}}$ *give continuous maps when instantiated and are natural transformations.*

The Radon monad is also strong, and, in fact, commutative. The left strength is given by the natural transformation $l_{\mathcal{R}} \colon X \times \mathcal{R}(Y) \to \mathcal{R}(X \times Y)$ *which takes* (x, μ_Y) *to the product measure* $\delta_x \times \mu_Y \colon A_X \times A_Y \mapsto \delta_x(A_X)\mu_Y(A_Y)$*. The resulting mediator* $\nabla_{\mathcal{R}} \colon \mathcal{R}(X) \times \mathcal{R}(Y) \to \mathcal{R}(X \times Y)$ *is exactly the construction of product measures,* $(\mu_X, \mu_Y) \mapsto \mu_X \times \mu_Y$*. The fact that the marginals of a product measure is equal to the measures of which the product was taken says that* $\mathcal{R}$ *is affine.*

The topology is an example of the weak-* topology, or the vague topology. It is generated by the sets

$$U_{f,\Omega} = \left\{ \mu \in \mathcal{R}(X) \mid \int_X f \,\mathrm{d}\mu \in \Omega \right\} \tag{2.25}$$

for $\Omega \subset \mathbb{C}$ open, and $f \colon X \to \mathbb{C}$ continuous. $\mathbb{R}$ and $[0,1]$ could both equivalently be used in place of $\mathbb{C}$ here and above.

Intuitively, as with the other probability monads, the multiplication says that from a random measure on X, one can form a non-random measure on X by averaging over the outcomes of that random measure. For example, in the case that $X = \mathbf{2} := \{H, T\}$, the discrete space on two elements, a random measure Φ is a random bias for a coin, a bag of variously biases coins. The measure $\mathrm{m}_{\mathcal{R}}(\Phi)$ then is given by the likelihood of flipping heads on a randomly drawn coin from the bag.

$\mathrm{m}_{\mathcal{R}}(\Phi)$ is also called *the barycenter of* Φ, results about which are discussed in greater detail in section 4.2.1 (see [17]). It is closely related to the idea of the *intensity measure* of a random measure [103, Lma. 2.4] in the theory of random measures.

Definition 2.28 (The Kleisli Category of the Radon Monad)**.** *Following definition 2.7,* the Kleisli category of the Radon monad, $\mathcal{K}l(\mathcal{R})$ *has compact Hausdorff spaces as its objects, and a morphism of this category from a space* X *to a space* Y*, notated with the Kleisli arrow* $X \rightsquigarrow Y$*, is exactly a continuous map* $X \to \mathcal{R}(Y)$*. Composition is Kleisli composition.*

Morphisms of the base category also exist in the Kleisli category: if $f\colon X \to Y$ is a morphism in **CH**, then the corresponding morphism in $\mathcal{K}l(\mathcal{R})$ is $\eta_Y \circ f\colon X \rightsquigarrow Y$. We call such a morphism *deterministic*.

A morphism $f\colon X \rightsquigarrow Y$ may equivalently be understood as an example of a *Markov kernel* in **Meas**: a map $X \times B(Y) \to [0,1]$ fulfilling various measurability conditions. As such the following notation is often useful: instead of writing $f(x)(A) \in [0,1]$ to represent the measure of a set $A \in B(Y)$ when using the measure $f(x) \in \mathcal{R}(Y)$ for a point $x \in X$ instead we write

$$f\left(A \,|\, x\right). \tag{2.26}$$

We write $f\left(- \,|\, x\right)$ for the measure $f(x) \in \mathcal{R}(Y)$ and integration of a function $\phi\colon Y \to \mathbb{C}$ with respect to this measure may be written as $\int_Y \phi(y)\, f\left(\mathrm{d}y \,|\, x\right)$.

The category $\mathcal{K}l(\mathcal{R})$ is the category of compact Hausdorff spaces and probabilistic processes between them. Kleisli composition acts as one would expect the composition of probabilistic processes to operate: if one has a method for using an element of X to generate a random element of Y, and one has a method for using an element of Y to generate a random element of Z, then given an element of X, first generate an element of Y and use this to generate a random element of Z. This is *averaging over* Y.

Explicitly, given $f\colon X \rightsquigarrow Y$ and $g\colon Y \rightsquigarrow Z$, the composition $g \circ f\colon X \rightsquigarrow Z$ is given by

$$g \circ f\left(A \,|\, x\right) = \int_{y \in Y} g\left(A \,|\, y\right)\, f\left(\mathrm{d}y \,|\, x\right). \tag{2.27}$$

Since the Radon monad is a monad, we may study its category of algebras. These have a familiar form.

Theorem 2.29 (Algebras of the Radon monad, [140, 150])**.** *The Eilenberg-Moore category of the Radon monad, $\mathcal{E}m(\mathcal{R})$, is equivalent to the category of convex, compact Hausdorff subspaces of locally convex topological spaces,* **ConvCH**.

We will make great use of this equivalence.

2.6 Kolmogorov Extension Theorem

Finally, we introduce a theorem that we will use throughout the work. The Kolmogorov extension theorem allows the construction of measures on infinite products of spaces by considering their marginals on finite products [For an example from Kolmogorov himself see 109, Sec. III.4]. There are a number of similar and equivalent forms. The below is from Bogachev [16, Thm. 7.7.1], specialised to **CH**.

Let J be a set and let $(X_j)_{j\in J}$ be an indexed set of compact Hausdorff spaces. By Tychonoff's theorem, the topological space $\prod_{j\in J} X_j$ with the product topology is also compact. It is also Hausdorff, and thus in **CH**.

Theorem 2.30 (Kolmogorov Extension Theorem)**.** *Suppose for every finite set* $I \subset J$*, there is a measure* $\mu_I \in \mathcal{R}(\prod_{i\in I} X_i)$*, with the property that if* $I_1 \subset I_2$ *then* $\mu_{I_1} = \pi_* \mu_{I_2}$*, for the projection* $\pi : \prod_{i\in I_2} X_i \to \prod_{i\in I_1} X_i$*.*

Then there exists a measure $\mu \in \mathcal{R}\Big(\prod_{j\in J} X_j\Big)$ *such that* $\mu_I = p_* \mu$ *for any projection* $p\colon \prod_{j\in J} X_j \to \prod_{i\in I} X_i$ *onto finitely many factors.*

It is also clear that such a measure is uniquely determined: if there are two such measures, they must differ on a cylinder set (these are generating sets for the topology on $\prod_{i\in I} X_i$and are of the form $\prod_{i\in J} A_i$, for open $A_i \subset X_i$ and $A_i \neq X_i$ for only finitely many indices) which would mean they would differ on some μ_I.

Since in all the work that follows we are concerned with countably infinite products of a single space, we shall use the following specialisation to the space $X^{\mathbb{N}}$ equipped with the product topology:

Theorem 2.31 (Kolmogorov Extension Theorem)**.** *Suppose for every* $n \in \mathbb{N}$*, there is a measure* $\mu_n \in \mathcal{R}(X^n)$*, with the property that* $\mu_n = \pi_* \mu_{n+1}$ *for* π *the projection* $\pi\colon\ (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$

Then there exists a measure $\mu \in \mathcal{R}\Big(X^{\mathbb{N}}\Big)$ *such that, for all* $n \in \mathbb{N}$*,* $\mu_n = (p_n)_* \mu$ *for* $p_n(x_i)_{i\in\mathbb{N}} = (x_1, \ldots, x_n)$*.*

This follows from theorem 2.30 by marginalising μ_n to fill in all other finite products.

3

Quantum de Finetti Theorems as Categorical Limits

Contents

3.1 Introduction

The main result of this chapter is theorem 3.72, the categorical quantum de Finetti theorem. This incorporates two traditional quantum de Finetti theorems, laid out in section 3.4, with different tensor products. This result is an essential stepping stone towards the categorical classical de Finetti theorems of chapter 4, and the coalgebraic reformulation of chapter 5. The chapter introduces all the necessary background for the use of C*-algebras, including the relationship between categories of C*-algebras and the Radon monad. The main theorem is then proved by exploiting this relationship and showing further properties of the state-space functor.

These results were originally presented in a paper at *QPL* 2022 and published in *EPTCS* 2023, co-authored with my supervisor, Prof. Sam Staton [144]. That paper contains all the auxiliary results, but only considered one tensor product. In this work, the main addition is that a second tensor product is incorporated, and proofs are laid out in greater detail.

3.1.1 De Finetti Theorems

The exchangeability theorems of chapters 3 to 5 are all de Finetti theorems, named for Italian mathematician Bruno de Finetti. In his 1937 paper *Foresight: Its Logical Laws, Its Subjective Sources* [34], De Finetti is interested in the problem of prediction under his subjectivist foundations for Probability Theory. Within de Finetti's subjectivism, the probability distributions of events are judgements of comparative likelihood, and do not approximate *real* quantities. Compare this with a frequentist interpretation of probability, which gives meaning to the probability of an event using the limiting frequency of this event in repeat experiments; a value the frequentist argues *is* 'real' and about which one can be correct or incorrect, near to or far from in approximation. To de Finetti, as long as a judgement of distributions cannot be arbitraged, which amounts to saying that assigned probabilities are additive and that the total probability is equal to 1, they cannot be *wrong*.

This basis of probability gives a clear definition of what probability *is*, but does so at the cost of constructs that were by that point already essential to how probability theory was being applied. One such modelling construct is a sequence of independent and identically distributed (i.i.d.) random variables. Modelling a sequence of real-world processes as an i.i.d. sequence $X_1, X_2, X_3, \ldots$ according to some unknown distribution μ allows data from those experiments to infer information about μ. For example, Rose might suppose that a coin-flip has a true bias towards heads of $p \in [0,1]$, of which she has limited knowledge. Calculating the sample mean $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \{\text{Flip } i \text{ is heads}\}$, she expects that $\mu_n \to p$ as $n \to \infty$. To de Finetti, this hidden distribution is an impossible construct, since distributions were themselves just subjective judgements of likelihood and to judge different trials as having some kind of identical underlying process needed significant justification unavailable from a real world process (what property is it that is shared by the identical repeated flips of a coin?). He writes: "an event is always a singular fact".

Foresight is concerned with what reasonable assumptions someone forming probabilistic judgements *can* make. In order to resurrect inference, he reasons about sequences of random variables with joint laws that are assumed invariant under the permutation of the underlying random variables. This is a property that he refers to as *exchangeability*, and he deems it a reasonable assumption that an observer of a process might make, based on their own judgement that they are unable to distinguish events under reordering.

The theorem from *Foresight* to which de Finetti gives his name shows that exchangeable sequences of random variables can always be decomposed into a mixture of independent and identically distributed sequences. In other words, given an exchangeable sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$, there is a **distribution Φ on distributions on** $\mathbb{R}$, such that the sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ generated by choosing a distribution μ at random using Φ and then producing an independent and identically μ-distributed sequence, is equal in distribution to $(X_n)_{n\in\mathbb{N}}$. In this way, we do not quite resurrect i.i.d. sequences, but we do at least regain the ability to reason about something: namely the (derived) distribution Φ, which we can approximate using data about the values that the X_ns take. With this, inference is restored, to de Finetti's philosophical relief.

Intuitively, this idea is well-suited to being described with category theory. The *mixing* of i.i.d. random variables as described above is just a composition of probabilistic processes as described in the Kleisli categories of a probability monad. The presence of a unique decomposition then hints at the construction of a limit in the category of such probabilistic process. The subject of this chapter and chapter 4 is exactly the translation of such theorems classifying exchangeable sequences of random objects into statements about limits of appropriate diagrams in appropriate categories of processes.

Since 1937, de Finetti's paper has inspired a healthy conversation about the contexts for generalisation and where similarly motivated theorems about classifying exchangeability can arise. Chapter 4 discusses classical de Finetti theorems: theorems about exchangeable sequences of random measures, which mostly resemble de Finetti's original theorem in form and conclusion. The current chapter strays from the setting of classical probability to instead discuss quantum de Finetti theorems: theorems classifying exchangeable sequences of quantum states.

Where de Finetti's theorem says that every exchangeable sequence of random variables can be decomposed into a mixture of independent and identically distributed random variables, quantum de Finetti theorems describe the decomposition of exchangeable sequences of quantum states into mixtures of infinite product states. They can be used to model an assumption about repeated testing of a quantum experiment and provides a correspondence between these and classical distributions over a single quantum state. In other words, we have a correspondence between repeat experiments and beliefs about the single state that is supposedly being tested.

This idea has, in the spirit of de Finetti's paper, inspired philosophical discussion about the foundation of quantum mechanics, particularly for subjectivist, or "quantum Bayesian", foundations, particularly those of Caves, Fuchs and Schack [23, 55]. Similarly to de Finetti, quantum Bayesianism (QBism) attempts finds an answer to the question of 'what is a quantum state?' in the subjective judgement of an observer. As such, inference poses a similar problem for the QBist as it does to the subjectivist, and is similarly reconciled with the quantum de Finetti theorem. This field of foundations is still live: see, for example, recent work of De Brota, Fuchs and Schack [35].

The original quantum de Finetti theorem is from Erling Størmer's 1969 paper *Symmetric States of Infinite Tensor Products of C*-algebras* and classifies exchangeable sequences of quantum states as (classical) mixtures of product states. Around the same time, Hulanicki and Phelps also published a functional-analytic result with an alternative statement of a quantum de Finetti theorem [80]. The Hulanicki-Phelps result differs from Størmer's in its choice of C*-tensor product; Landsman also makes this alternative choice in the quantum de Finetti theorem of his book [114], drawing his proof from Hudson and Moody [79].

The categorical limit in the quantum case is not as easy to spot as in the classical situation, since there is no monad of quantum processes. Nonetheless, a categorical approach is still possible.

An interesting aspect of these theorems is that the classification uses a combination of classical probability and quantum randomness. Tools of operator algebra, particularly the theory of C*-algebras, act as a unifying setting where these can mingle.

The relationships are elucidated in the categorical picture which reveals quantum randomness in the Eilenberg-Moore algebras of a probability monad. In this setting, it is also clearer how the quantum picture and the classical picture of classifying exchangeability align. Since categories of C*-algebras are non-commutative duals to those of probabilistic processes, in these categories we work with colimits.

The results of this chapter were first published in [144], the main result of which is the half of theorem 3.72 referring to the minimal tensor product. Since then the discovery of Hulanicki and Phelps' paper has allowed us to elaborate on our result, extending it to the maximal tensor product too.

The results in chapter 4 are all proved via the correspondences between quantum probability and classical probability, so it cannot be read without the content of this chapter. To accommodate for a reader that is unfamiliar with quantum probability and is primarily interested in the classical case, this chapter has been written to be readable for those without a background in quantum probability. Since the results in the chapter are primarily about operator algebras, attempts have been made to also bridge this with the traditional Hilbert space formalisation of quantum physics, though this author is neither a physicist nor a functional analyst so for a more comprehensive approach please see Landsman's extensive *Foundations of Quantum Theory* [114].

Sourcing definitions and propositions We continue to use Emily Riehl's book [131] as the source for category theory definitions. Landsman is our primary source for results and definitions about quantum foundations and operator algebras, all results with a label but without a citation can be found in [114].

3.1.2 The Main Result

The outline here was first given in a similar way in our paper [144]. Let $\mathcal{H}$ be a Hilbert space. A *sequence of states* on $\mathcal{H}$ is a collection of quantum states on the tensor powers of $\mathcal{H}$: a state ρ_1 for $\mathcal{H}^{\otimes 1} := \mathcal{H}$, a state ρ_2 for $\mathcal{H}^{\otimes 2} := \mathcal{H} \otimes \mathcal{H}$, and so on, with a state ρ_n for $\mathcal{H}^{\otimes n} := \bigotimes_{i=1}^{n} \mathcal{H}$ for each $n \in \mathbb{N}$.

In the case that $\mathcal{H} = \mathbb{C}^2$, this is a sequence of states on increasing numbers of qubits, and is mathematically expressed as a sequence of density matrices in $\mathbb{C}^{2^n \times 2^n}$.

We may ask questions about the relationships between these states. For example, we may move from states on $\mathcal{H}^{\otimes n}$ to states on $\mathcal{H}^{\otimes m}$, for $m \leq n$, by taking the partial trace over the last $m - n$ factors in the tensor product. If a sequence of states $(\rho_n)_{n \in \mathbb{N}}$ is such that for any choices of natural numbers $m \leq n$, then ρ_m is the partial trace of ρ_n then we call the sequence *consistent*. A consistent sequence of states should invoke the image of large but finite preparations of quantum experiments, where we may choose to prepare larger and larger numbers of e.g. qubits, though we may also discard later preparations to get back to where we were.

A further limiting condition for a sequence like this is exchangeability. A consistent sequence of states $(\rho_n)_{n \in \mathbb{N}}$ is called *exchangeable* if each ρ_n is invariant under permuting the underlying factor Hilbert spaces $\mathcal{H}$. For example, for ρ_2, this would mean that $\rho_2 = \rho_2 \circ swap$ for $swap : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ the unique map defined by $swap(a \otimes b) = b \otimes a$.

In a categorical context, a useful way of understanding a quantum state on a Hilbert space $\mathcal{H}$ is as a quantum channel $\mathbb{C} \to \mathcal{H}$. This amounts to a completely-positive trace-preserving map between the corresponding spaces of density matrices. In that case, a sequence of states looks like maps of the following form:

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[dll, "\rho_1"'] \arrow[dl, "\rho_2"] \arrow[d, "\rho_3"] \arrow[drr, "\rho_n"] & & & \\
\mathcal{H} & \mathcal{H}^{\otimes 2} & \mathcal{H}^{\otimes 3} & \dots & \mathcal{H}^{\otimes n} & \dots
\end{tikzcd} \tag{3.1}$$

Consistency and exchangeability then can be phrased as in the language of cones over diagrams.

For consistency, the diagram is $(\mathbb{N}, \leq)^{\mathrm{op}}$-shaped, taking n to $\mathcal{H}^{\otimes n}$ and for each $m \leq n$, the map $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m}$ is given by the partial trace over the last $n - m$ components. These maps are generated by the partial traces over the final component $\mathrm{tr}_n \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$.

$$\begin{tikzcd}
\mathcal{H} & \mathcal{H}^{\otimes 2} \arrow[l, "\mathrm{tr}_2"'] & \dots \arrow[l, "\mathrm{tr}_3"'] & \mathcal{H}^{\otimes n} \arrow[l, "\mathrm{tr}_n"'] & \dots . \arrow[l, "\mathrm{tr}_{n+1}"']
\end{tikzcd} \tag{3.2}$$

A sequence of states as in diag. (3.1) is consistent if it forms a cone of diag. (3.2).

A sequence is exchangeable if it forms a cone over the $\mathbf{Inj}^{\mathrm{op}}$-shaped diagram that takes n to $\mathcal{H}^{\otimes n}$, a permutation $\sigma \in \mathbf{Inj}(n, n)$ to the respective permutation of tensor factors $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$, and the identity injection $\{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ to the partial trace over the last $n - m$ factors, $\mathrm{tr}_{nm} \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m}$. All other morphisms in **Inj** are generated from these.

$$\begin{tikzcd}
\mathcal{H} \arrow[loop above] & \mathcal{H}^{\otimes 2} \arrow[l, shift right, "\mathrm{tr}_2"'] \arrow[l, shift left] \arrow[loop above, "swap"] \arrow[loop below] & \dots \arrow[l, shift right=5] \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \arrow[l, shift left=5] & \mathcal{H}^{\otimes n} \arrow[l, shift right=3] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=3] \arrow[loop above] \arrow[loop below] & \dots \arrow[l, shift right=3] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=3]
\end{tikzcd} \tag{3.3}$$

The main result of [144], given here in theorem 3.72, is showing that this diagram has a limit, and this limit gives a universal property for the state space of a quantum system.

This theorem is built on the back of Størmer's quantum de Finetti theorem [148], which decomposes exchangeable sequences of states into mixtures of infinite product states. A categorical description of this theorem extends in three steps:

1. First, we note that Størmer's theorem results in a classical mixture of product states, which is to say a measure on the space of states on an individual system. As such it is necessary to use categories that include both quantum and classical randomness. For this we use the category $\mathbf{CSt}_{\mathrm{CPU}}^{\mathrm{op}}$, the opposite of the category C*-algebras and completely positive, unital maps between them. Both the category of quantum channels and the Kleisli category of the Radon probability monad embed in this category, and it presents a convenient setting for a quantum de Finetti theorem.

2. Secondly, we extend from the classification of exchangeable sequences above as cones with apex $\mathbb{C}$ to *parametrised* exchangeable sequences: cones $\mathcal{K} \to \mathcal{H}^{\otimes n}$ over the exchangeability diagram parametrised by some space $\mathcal{K}$ incorporating both classical and quantum randomness.

$$\begin{tikzcd}
\mathcal{H} \arrow[loop above] & \mathcal{H}^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[loop, in=120, out=60, distance=2em] \arrow[loop, in=150, out=90, distance=2em] & \mathcal{H}^{\otimes 3} \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop, in=150, out=110, distance=2em] \arrow[loop, in=130, out=90, distance=2em] \arrow[loop, in=110, out=70, distance=2em] \arrow[loop, in=90, out=50, distance=2em] \arrow[loop, in=70, out=30, distance=2em] \arrow[loop, in=50, out=10, distance=2em] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \mathcal{H}^{\otimes n} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \arrow[loop, in=150, out=110, distance=2em] \arrow[loop, in=130, out=90, distance=2em] \arrow[loop, in=110, out=70, distance=2em] \arrow[loop, in=90, out=50, distance=2em] \arrow[loop, in=70, out=30, distance=2em] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \\
 & & \mathcal{K} \arrow[ull, bend left] \arrow[ul, bend left] \arrow[u] \arrow[urr, bend right] & & &
\end{tikzcd} \tag{3.4}$$

3. In this context we may ask whether there exists a C*-algebra, $\mathrm{QdF}(\mathcal{H})$ and a cone over the exchangeability diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[loop above] & \mathcal{H}^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[loop, in=120, out=60, distance=2em] \arrow[loop, in=150, out=90, distance=2em] & \mathcal{H}^{\otimes 3} \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop, in=150, out=110, distance=2em] \arrow[loop, in=130, out=90, distance=2em] \arrow[loop, in=110, out=70, distance=2em] \arrow[loop, in=90, out=50, distance=2em] \arrow[loop, in=70, out=30, distance=2em] \arrow[loop, in=50, out=10, distance=2em] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \mathcal{H}^{\otimes n} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \arrow[loop, in=150, out=110, distance=2em] \arrow[loop, in=130, out=90, distance=2em] \arrow[loop, in=110, out=70, distance=2em] \arrow[loop, in=90, out=50, distance=2em] \arrow[loop, in=70, out=30, distance=2em] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \\
 & & & & & \mathrm{QdF}(\mathcal{H}) \arrow[ulllll, bend left] \arrow[ullll, bend left] \arrow[ulll, bend left] \arrow[ul, bend left]
\end{tikzcd} \tag{3.5}$$

   which is the limit in $\mathbf{CSt}_{\mathrm{CPU}}^{\mathrm{op}}$. That is to say this cone is such that, given an exchangeable sequence parametrised by another C*-algebra $\mathcal{K}$, there is a unique factorisation of this sequence through the cone over $\mathrm{QdF}(\mathcal{H})$: a parametrised exchangeable sequence of states is equivalent to a single channel of quantum and classical information to $\mathrm{QdF}(\mathcal{H})$.

$$\begin{tikzcd}
\mathcal{H} \arrow[loop above] & \mathcal{H}^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above, out=60, in=30] & \mathcal{H}^{\otimes 3} \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop above, out=150, in=120] \arrow[loop above, out=125, in=100] \arrow[loop above, out=105, in=85] \arrow[loop above, out=95, in=75] \arrow[loop above, out=80, in=55] \arrow[loop above, out=60, in=30] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \mathcal{H}^{\otimes n} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \arrow[loop above, out=150, in=120] \arrow[loop above, out=125, in=100] \arrow[loop above, out=105, in=85] \arrow[loop above, out=95, in=75] \arrow[loop above, out=80, in=55] \arrow[loop above, out=60, in=30] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \\
 & & & & & \mathrm{QdF}(\mathcal{H}) \arrow[ulllll, bend left] \arrow[ullll, bend left] \arrow[ulll, bend left] \arrow[ul, bend left] \\
 & & \mathcal{K} \arrow[uull, bend left] \arrow[uul, bend left] \arrow[uu] \arrow[uurr, bend right] \arrow[urrr, dashed, bend right]
\end{tikzcd} \tag{3.6}$$

Theorem 3.72 states that such a limit does exist. Guided by Størmer, we showed that the right choice for this C*-algebra is the space corresponding to classical distributions over all states on $\mathcal{H}$ itself.

The unparametrised quantum de Finetti theorems allow us to construct the mediating morphism of the factorisation diag. (3.6) pointwise, considering the state at a given value of the parameter. The question then is whether such a point-wise defined map is a morphism in $\mathbf{CSt}_{\mathrm{CPU}}$. The proof that this is indeed a limit is fundamentally categorical. It relies on the relationships between states on C*-algebras and classical distributions on compact Hausdorff spaces, as described by the Radon monad and its monad algebras.

3.2 Staging C*-Algebras

All definitions and labelled propositions may be found in Landsman [114]. Where definitions are drawn from other sources, they are cited inline.

To introduce the Quantum de Finetti theorems of sections 3.4 and 3.5, the technical background for a C*-algebra approach to quantum processes must be laid out. This section is intended to lead a probabilistically-informed reader naturally through new material such that the main theorems of this chapter feel like natural formulations. Attention is paid here to extricating the details of definitions and giving examples to assist such a probability theorist familiarising themselves with the concepts introduced. We take a limited approach to the theory of operator algebras, and introduce exactly enough in the way of definitions and propositions and no more to set the stage for the results of section 3.5.

3.2.1 C*-Algebras

Definition 3.1 (Banach Algebra, *-Algebra, C*-Algebra)**.** *An* algebra *is a vector space V with a bilinear binary operator $V \times V \to V$, notated on elements by*

$$(u, v) \mapsto uv \text{ or } u \cdot v \tag{3.7}$$

called multiplication*. If this operation is commutative, V is said to be a* commutative algebra*. If there exists an element $1_V \in V$ which acts as a unit for multiplication, then V is called* unital*.*

A normed algebra V over $\mathbb{C}$ with norm $\|\cdot\|_V$ is called a Banach algebra *if V is complete with respect to $\|\cdot\|_V$ and the norm is submultiplicative: for all $u, v \in V$,*

$$\|uv\| \leq \|u\|\|v\|. \tag{3.8}$$

We will assume further that all Banach algebras are unital and have a multiplicative unit $1_V \in V$.

A *-algebra *is an algebra $(\mathcal{A}, \cdot)$ with an operation $(-)^* : X \to X$ called* involution *which is self-inverse, conjugate-linear and anti-multiplicative:*

$$(xy)^* = y^* x^*. \tag{3.9}$$

It is called a Banach *-algebra *if the underlying algebra is given the structure of a Banach algebra.*

*A Banach *-algebra $(\mathcal{A}, \|\cdot\|, \cdot, (-)^*)$ which satisfies the* C*-identity

$$\|x^* x\| = \|x\|^2 \tag{3.10}$$

for all $x \in X$ is called a C*-algebra*.*

The C*-identity, which can be expressed in a number of equivalent ways depending on which products of x and x^* we take under the norm, is very strong, as is displayed in the classification theorems later in this section (e.g. theorem 3.10). A Banach *-algebra has both algebraic properties as an algebra, and analytic properties as a Banach space. The C*-identity is exactly the property needed to force that these two sides of a *-algebra $\mathcal{A}$ play well together. In particular, a Banach algebra admits at most one C*-norm (for one justification of Jacobs and Furber [58, p4] with the use of [99, Prop. 4.1.1(i)], though the idea is much older. The sketch is that the spectral radius of self-adjoint elements is purely algebraic, whilst the C*-identity links such radii to the norm of a general element).

Note that the C*-identity also immediately implies that involution is a continuous operation:

Proposition 3.2. *For a C*-algebra $\mathcal{A}$, involution $(-)^* \colon \mathcal{A} \to \mathcal{A}$ is an isometry. In particular, it is continuous.*

Proof. For $x \in \mathcal{A}$ we have $\|x^*x\| = \|x\|^2$. By submultiplicativity of the norm we also have $\|n^*x\| \leq \|x\|\|x^*\|$. Thus, $\|x\| \leq \|x^*\|$. This also gives us $\|x^*\| \leq \|x^{**}\| = \|x\|$ and so $\|x\| = \|x^*\|$. ☐

Though some results used in what follows have non-unital versions, the original research in this work requires the use of unital C*-algebras. It would be an interesting avenue of future work to explore whether non-unital analogues are also true.

Let us look at some examples of C*-algebras.

Example 3.3. *For any natural number $n \in \mathbb{N}$, $\mathbb{C}^n$ equipped with the Euclidean norm and component-wise multiplication and conjugation is a commutative C*-algebra.*

Example 3.4. *For any natural number $n \in \mathbb{N}$, let $M^n(\mathbb{C})$ be the space of complex-valued $n \times n$ matrices. Given the standard matrix multiplication, the identity matrix as unit and Hermitian transposition as involution, and equipped with the operator norm, this is a C*-algebra.*

We refrain from using the notation here $\mathbb{C}^{n\times n}$ to avoid confusion with the commutative C*-algebra $\mathbb{C}^{\left(n^2\right)}$ of n^2-tuples of complex numbers with pointwise multiplication.

Embracing *-algebras as algebraic objects suggests defining morphisms between them as functions preserving the algebraic operations.

Definition 3.5 (*-homomorphisms)**.** *A linear map between *-algebras $\phi\colon \mathcal{A} \to \mathcal{B}$ is* unital *if it preserves the unit of $\mathcal{A}$: $\phi(1_A) = 1_B$.*

A *-homomorphism *is a unital linear map between *-algebras $\phi\colon \mathcal{A} \to \mathcal{B}$ which preserves multiplication and involution: for all $a_1, a_2 \in A$,*

$$\phi(a_1^*) = \phi(a_1)^* \text{ and } \phi(a_1 a_2) = \phi(a_1)\phi(a_2). \tag{3.11}$$

The category of all C-algebras and *-homomorphisms is notated with $\mathbf{CSt}_{\mathrm{MIU}}$.*

The full subcategory $\mathbf{CSt}_{\mathrm{MIU}}$ consisting only of the commutative C-algebras, is notated by $\mathbf{cCSt}_{\mathrm{MIU}}$.*

**-homomorphisms are linear maps of normed vector spaces, so we may consider their topological properties. A *-homomorphism which is also an isometry of normed vector spaces is called a* *-isometry*.*

Proposition 3.6 (Russo-Dye Theorem [133, Cor. 1])**.** *All *-homomorphisms, when considered as linear maps of normed spaces, have unit norm.*

As such, there is a forgetful functor $\mathbf{CSt}_{\mathrm{MIU}} \to \mathbf{Ban}$, the category of Banach spaces and short linear maps, i.e. maps with norm less than or equal to one.

All finite-dimensional C*-algebras are isomorphic in $\mathbf{CSt}_{\mathrm{MIU}}$ to a finite direct sum as Banach spaces, with component-wise operations, of $M^{n_i}(\mathbb{C})$ for some finite sequence of natural numbers n_i [e.g. see 152, Thm. 11.3]. For example,

$$\mathbb{C}^n \cong \bigoplus_{i=1}^{n} \mathbb{C} = \bigoplus_{i=1}^{n} M^1(\mathbb{C}). \tag{3.12}$$

$M^n(\mathbb{C})$ is isomorphic to the space of bounded operators on the finite-dimensional Hilbert space $\mathbb{C}^n$. The C*-algebra structure naturally arises from this picture and generalises to any Hilbert space $\mathcal{H}$.

Definition 3.7. *Let $\mathcal{H}$ be a Hilbert space. Define*

$$\mathcal{B}(\mathcal{H}) := \{\Phi\colon \mathcal{H} \to \mathcal{H} \mid \Phi \text{ is linear and bounded}\}. \tag{3.13}$$

$\mathcal{B}(\mathcal{H})$ is a Banach space when equipped with the operator norm

$$\|\Phi\|_{\mathcal{B}(\mathcal{H})} := \|\Phi\|_{\mathrm{op}} = \sup_{v\in\mathcal{H}\setminus\{0\}} \frac{\|\Phi(v)\|}{\|v\|} \tag{3.14}$$

and under this norm, is an algebra with composition

$$\Phi \circ \Psi\colon v \mapsto \Phi\left(\Psi\left(v\right)\right) \tag{3.15}$$

as multiplication. With the identity operator $\mathrm{id}_{\mathcal{H}}\colon v \mapsto v$ as unit, and taking the adjoint of an operator as an involution $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$ is a C-algebra.*

Such spaces are mostly highly non-commutative, even for simple Hilbert spaces, as seen above for $\mathcal{H} = \mathbb{C}^n$. An important class of commutative algebras are born from topology.

Definition 3.8. *Let $X \in \mathbf{CH}$ be a compact Hausdorff topological space (definition 2.15). Consider the set $\mathbf{CH}(X,\mathbb{C}) = \{\phi\colon X \to \mathbb{C} \mid \phi \text{ is continuous}\}$.*

Since X is compact, the supremum norm is well-defined on $\mathbf{CH}(X,\mathbb{C})$:

$$\|f\|_{\sup} := \sup_{x\in X} |f(x)| < \infty. \tag{3.16}$$

If we define multiplication pointwise, that is $f \cdot g(x) = f(x)g(x)$; involution as pointwise complex conjugation; and the unit as the constant map $1_X\colon x \mapsto 1$, $\mathbf{CH}(X,\mathbb{C})$ has the structure of a C-algebra. We will write this C*-algebra as $C(X) := \mathbf{CH}(X,\mathbb{C})$.*

*Note that for $X, Y \in \mathbf{CH}$, $\phi \in C(X)$ and continuous $f\colon X \to Y$, the map $\mathbf{CH}(f,\mathbb{C})(\phi) := f^*(\phi) = \phi \circ f$ is also continuous. This gives a *-homomorphism*

$$C(f) := f^*\colon C(Y) \to C(X). \tag{3.17}$$

As such, $C\colon \mathbf{CH} \to \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{MIU}}$ is a functor. We have that $U_{\mathbf{CSt}_{\mathrm{MIU}}} \circ C = \mathbf{CH}(-,\mathbb{C})$, for the forgetful functor $U_{\mathbf{CSt}_{\mathrm{MIU}}}\colon \mathbf{CSt}_{\mathrm{MIU}} \to \mathbf{Set}$ sending a C-algebra to its set of points.*

Example 3.9. *If X is the discrete space on n points, then $C(X) \cong \mathbb{C}^n$. Given any enumeration $\{x_1, \dots, x_n\}$ of elements of X, one isomorphism takes $\phi \in C(X)$ to $(\phi(x_1), \dots, \phi(x_n))$.*

In fact, a famous result of Gelfand's notes that every commutative C*-algebra can be realised in this way:

Theorem 3.10 (Gelfand Duality, Thm. C.23)**.** *Let $\mathcal{A} \in \mathbf{cCSt}_{\mathrm{MIU}}$ be a commutative C*-algebra (with a unit). There exists a compact Hausdorff space $X \in \mathbf{CH}$ such that $\mathcal{A} \cong C(X)$ in $\mathbf{CSt}_{\mathrm{MIU}}$.*

In other words, C is essentially surjective. It is one leg of the equivalence of categories $\mathbf{CH} \cong \mathbf{cCSt}_{\mathrm{MIU}}^{\mathrm{op}}$.

Since the study of commutative C*-algebras is then reduced (or extended) to studying the properties of a class of topological spaces, the study of non-commutative C*-algebras is sometimes called *non-commutative geometry.*

Theorem 3.11 (Gelfand-Naimark Theorem, Thm. C.87)**.** *Let $\mathcal{A} \in \mathbf{CSt}_{\mathrm{MIU}}$ be any C*-algebra. Then there exists a Hilbert space $\mathcal{H}_{\mathcal{A}}$ and a *-isometry $\mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{A}})$.*

That is to say: every C*-algebra is isometrically *-isomorphic to a subalgebra of operators on a Hilbert space.

Recall a category $\mathbf{C}$ is called *complete* if every diagram from a small category $\mathcal{I}$ (that is, $\mathrm{ob}(\mathcal{I})$ is a set) to $\mathbf{C}$ has a limit. If the same is true for colimits instead, $\mathbf{C}$ is called *cocomplete.* $\mathbf{CSt}_{\mathrm{MIU}}$ is both.

Theorem 3.12 ([20, Cor. 7.23])**.** *The category $\mathbf{CSt}_{\mathrm{MIU}}$ is complete and cocomplete.*

3.2.2 Positivity, States and the State-space functor

Definition 3.13 (Positive Cone)**.** *An element of a *-algebra $a \in \mathcal{A}$ is called* positive *if it is self-adjoint, that is $a^* = a$, and its spectrum*

$$\mathrm{spec}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \cdot 1_{\mathcal{A}} \textit{ is non-invertible}\} \tag{3.18}$$

is contained in the non-negative reals $\mathbb{R}_{\geq 0} \subset \mathbb{C}$.

If a is positive, we write $0 \leq a$. If $0 \leq b - a$ for $a, b \in \mathcal{A}$, then we write $a \leq b$. This is a partial order on $\mathcal{A}$.

The set $\mathcal{A}_+ := \{a \in \mathcal{A} \mid 0 \leq a\}$ is called the positive cone *of $\mathcal{A}$. It is a* cone *in the precise sense that it is closed under addition and scalar multiplication by positive real numbers. $\mathcal{A}_+$ is not necessarily closed under multiplication.*

Proposition 3.14 ([152, p23])**.** *The following are equivalent for an element $a \in \mathcal{A}$ of a C^*-algebra:*

1. *$a \in \mathcal{A}_+$.*
2. *There exists $b \in \mathcal{A}$ such that $a = b^* b$.*
3. *There exist $\{b_1, b_2, \dots, b_n\} \subset \mathcal{A}$ such that $a = \sum_{i=1}^n b_i^* b_i$.*

Proposition 3.15 (Prop. C.51)**.** *The positive cone is convex.*

Proposition 3.16. *The positive cone is topologically closed.*

Proof. This follows from proposition 3.14 and continuity of both multiplication and involution. □

Example 3.17. *The positive elements of $\mathcal{B}(\mathcal{H})$, for finite-dimensional $\mathcal{H}$, are exactly the positive semi-definite operators.*

For some topological space X, a continuous function $\phi\colon X \to \mathbb{C}$ is of the form $\psi^*\psi = |\psi|^2$ for some $\psi \in C(X)$ if and only if $\phi(X) \subset \mathbb{R}_{\geq 0} \subset \mathbb{C}$. In the commutative finite-dimensional case of $\mathbb{C}^n$, the positive cone is exactly $\mathbb{R}^n_{\geq 0} \subset \mathbb{C}^n$.

All *-homomorphisms f preserve positive elements since $f(a^*a) = f(a)^* f(a)$. However, there are linear maps that also preserve these positive elements and are not be multiplicative.

Example 3.18. *We define the map $f_{\frac{1}{2}}\colon \mathbb{C}^2 \to \mathbb{C}$ given on the standard basis as*

$$\begin{aligned} (1,0) &\mapsto \frac{1}{2} \\ (0,1) &\mapsto \frac{1}{2}. \end{aligned} \tag{3.19}$$

This map is linear by definition and unital, since

$$f_{\frac{1}{2}}\left(1,1\right) = f_{\frac{1}{2}}\left(1,0\right) + f_{\frac{1}{2}}\left(0,1\right) = 1 \tag{3.20}$$

but is certainly not multiplicative:

$$f_{\frac{1}{2}}\left(0,0\right) = 0 \neq \frac{1}{4} = f_{\frac{1}{2}}\left(1,0\right) \cdot f_{\frac{1}{2}}\left(0,1\right). \tag{3.21}$$

It is, however, positive, since any pair of non-negative reals (r_1, r_2) *will be mapped to* $\frac{r_1}{2} + \frac{r_2}{2}$*, another non-negative real.*

Maps like these are essential for using C*-algebras both to model quantum channels and for introducing classical probability.

Definition 3.19 (Positive Maps of *-Algebras)**.** *Given two *-algebras* $\mathcal{A}, \mathcal{B}$*, a linear map* $\phi\colon \mathcal{A} \to \mathcal{B}$ *is* positive *if it takes positive elements in* $\mathcal{A}$ *to positive elements in* $\mathcal{B}$*.*

The category of C-algebras and positive, unital linear maps is notated by* $\mathbf{CSt}_{\mathrm{PU}}$*. If restricting to only commutative C*-algebras, we write* $\mathbf{cCSt}_{\mathrm{PU}}$*.*

We have the following diagram of inclusions of categories of C*-algebras:

$$\begin{tikzcd}
 & \mathbf{cCSt}_{\mathrm{PU}} \arrow[dr, two heads] & \\
\mathbf{cCSt}_{\mathrm{MIU}} \arrow[ur, tail] \arrow[dr, two heads] & & \mathbf{CSt}_{\mathrm{PU}} \\
 & \mathbf{CSt}_{\mathrm{MIU}} \arrow[ur, tail] &
\end{tikzcd} \tag{3.22}$$

Recall that tailed arrows are wide (they are bijections on objects), whilst double-headed functors are full and faithful (they are bijections on morphisms in their image).

Proposition 3.20 ([126, Prop. 2.1])**.** *All positive, unital maps have unit norm.*

Definition 3.21 (States on a C*-Algebra)**.** *Given a C*-algebra* $\mathcal{A}$*, a positive, unital map* $\rho\colon \mathcal{A} \to \mathbb{C}$ *is called* a state*.*

$$\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A}, \mathbb{C}) = \{\rho\colon \mathcal{A} \to \mathbb{C} \mid \rho \textit{ is a state}\} \tag{3.23}$$

is the set of states on $\mathcal{A}$*.*

Proposition 3.22 ([58])**.** *The set of states* $\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C})$ *of a C*-algebra* $\mathcal{A}$ *has the following properties:*

1. *With pointwise addition and scalar multiplication, it is convex.*
2. *Given the weak-* topology, it is compact and Hausdorff.*
3. *If* $\phi\colon \mathcal{A}\to\mathcal{B}$ *is a positive, unital map, the pre-composition map*

$$\mathbf{CSt}_{\mathrm{PU}}(\phi,\mathbb{C}) := -\circ\phi\colon \mathbf{CSt}_{\mathrm{PU}}(\mathcal{B},\mathbb{C})\to\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C}) \tag{3.24}$$

is affine and continuous.

Recall that the weak-* topology on $\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C})$ has the basis of sets of the form $V_{a,\Omega} := \{\rho\in\mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C})\mid \rho(a)\in\Omega\}$ where $a\in\mathcal{A}$ and $\Omega\subset\mathbb{C}$ is open. This is the coarsest topology that makes the evaluation maps

$$\begin{aligned}\mathrm{ev}_a\colon \mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C}) &\to \mathbb{C}\\ \rho &\mapsto \rho(a)\end{aligned} \tag{3.25}$$

continuous for all $a\in\mathcal{A}$.

Definition 3.23. *The* state-space of $\mathcal{A}$ *is* $S(\mathcal{A}) := \mathbf{CSt}_{\mathrm{PU}}(\mathcal{A},\mathbb{C})$ *with the weak-* topology and convex structure above.*

By proposition 3.22, $S\colon \mathbf{CSt}_{\mathrm{PU}}\to\mathbf{ConvCH}$ *is a functor.*

We have that $U_{\mathbf{CH}}\circ U_{\mathbf{ConvCH}}\circ S = \mathbf{CSt}_{\mathrm{PU}}(-,\mathbb{C})$:

Corollary 3.24. *The hom-functor* $\mathbf{CSt}_{\mathrm{PU}}(-,\mathbb{C})\colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}}\to\mathbf{Set}$ *factors through the state-space functor and the two monadic forgetful functors* $U_{\mathbf{ConvCH}}\colon \mathbf{ConvCH}\to\mathbf{CH}$ *and* $U_{\mathbf{CH}}\colon \mathbf{CH}\to\mathbf{Set}$.

As such, the notation S will be used for either of the functors from $\mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}}$ to $\mathbf{ConvCH}$ or $\mathbf{CH}$ depending on the context.

An insightful result of Jacobs and Furber says that the positive, unital maps between C*-algebras are equivalently given by the affine, continuous maps between their state-spaces.

Theorem 3.25 ([58, Thm. 5.21])**.** *The state-space functor* $S\colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{ConvCH}$ *is full and faithful.*

$$\begin{tikzcd}
\mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \arrow[r, two heads, "S"] \arrow[rrr, bend right, "\mathbf{CSt}_{\mathrm{PU}}(-{,}\mathbb{C})"'] & \mathbf{ConvCH} \arrow[r, "U_{\mathbf{ConvCH}}"] & \mathbf{CH} \arrow[r, "U_{\mathbf{CH}}"] & \mathbf{Set}
\end{tikzcd} \tag{3.26}$$

As such, there is a full subcategory of $\mathbf{ConvCH}^{\mathrm{op}}$ equivalent to $\mathbf{CSt}_{\mathrm{PU}}$. This subcategory is characterized by Alfsen and Shultz [6], who explore which cancellative, convex, compact Hausdorff spaces arise as the state spaces of C*-algebras. The classification is complicated and algebraic. We would be interested whether there is a simpler categorical classification.

3.2.3 Probabilistic Gelfand Duality

Let's consider a map $\phi\colon \mathbb{C}^n \to \mathbb{C}$ in $\mathbf{CSt}_{\mathrm{PU}}$. ϕ is defined on the basis vectors $e_i := (0,\dots,0,\underbrace{1}_{i\text{th}},0\dots,0)$, so we may consider necessary and sufficient conditions on the values $p_i = \phi(e_i)$ such that ϕ is a state.

Given $e_i = e_i^* e_i$ is positive, p_i must be a non-negative real number for each i. Further, since the C*-algebra unit $1_{\mathbb{C}^n} = \sum_{i=1}^n e_i$, we must have that

$$\sum_{i=1}^n p_i = \phi\left(\sum_{i=1}^n e_i\right) = \phi\left(1_{\mathbb{C}^n}\right) = 1. \tag{3.27}$$

This is to say that a positive, unital map $\mathbb{C}^n \to \mathbb{C}$ is exactly a probability distribution on the discrete space $\{1,\dots,n\}$. The positive, unital map $f_{\frac{1}{2}}\colon \mathbb{C}^2 \to \mathbb{C}$ of example 3.18, corresponds to a Bernoulli $\left(\frac{1}{2}\right)$ distribution on $\{0,1\}$. This is not a coincidence. By removing the restrictive condition of multiplicativity from our morphisms and asking instead only for positivity and preservation of unit, we have introduced probability into our category.

Recall the Radon monad on the category **CH** of compact Hausdorff topological spaces from definition 2.27. This monad takes a space X to the set $\mathcal{R}(X) = \{\mu\colon B(X) \to [0,1] \mid \mu \text{ is a Radon measure}\}$ with a basis of the topology given by sets of the form $U_{f,\Omega} = \{\mu \in \mathcal{R}(X) \,|\, \int_X f \, \mathrm{d}\mu \in \Omega\}$ for $\Omega \subset \mathbb{C}$ open, and $f \colon X \to \mathbb{C}$ continuous.

The above example tells us that $S(\mathbb{C}^n) \cong \mathcal{R}(\{1,\dots,n\})$. Recall that $\mathbb{C}^n \cong C(\{1,\dots,n\})$ and as such we have:

$$SC(\{1,\dots,n\}) \cong \mathcal{R}(\{1,\dots,n\}). \tag{3.28}$$

This is a specific case of a general fact:

Theorem 3.26 (Radon Monad as States)**.** *As endofunctors on* $\mathbf{CH}$*,* $\mathcal{R}$ *and* $S \circ C$ *are naturally isomorphic.*

Proof. This result is a categorical form of the Riesz-Markov-Kakutani representation theorem. For $X \in \mathbf{CH}$, we define $\alpha_X \colon \mathcal{R}(X) \to SC(X)$ by taking a measure $\mu \in \mathcal{R}(X)$ to the functional $\alpha_X(\mu) \colon C(X) \to \mathbb{C}$ given by

$$\alpha_X(\mu)(f) \to \int_X f \, \mathrm{d}\mu \tag{3.29}$$

for $f \in C(X)$.

The Riesz-Markov-Kakutani representation theorem guarantees that this map is both injective and surjective [See, for example, 100, Thm. 9].

Take a basis set $V = \{\rho \in SC(X) \mid \rho(f) \in \Omega\}$ in $SC(X)$ for some continuous $f \colon X \to \mathbb{C}$ and open $\Omega \subset \mathbb{C}$. Then $\alpha_X^{-1}(V) = \{\mu \in \mathcal{R}(X) \mid \alpha_X(\mu) \in V\} = \{\mu \in \mathcal{R}(X) \mid \int f \, \mathrm{d}\mu \in \Omega\}$ is exactly the basis open set $U_{f,\Omega} \subset \mathcal{R}(X)$.

Since all spaces involved are compact and Hausdorff, continuous bijections between them are homeomorphisms, and thus α_X is a homeomorphism as desired. ☐

$$\begin{tikzcd}
& \mathbf{CSt}_{\mathrm{PU}} \arrow[dl, "S"'] \arrow[d, phantom, "\perp"] & \\
\mathbf{ConvCH} \arrow[rr, "U"'] & {} & \mathbf{CH} \arrow[ul, "C"'] \arrow[loop right, "\mathcal{R}"]
\end{tikzcd} \tag{3.30}$$

So Radon measures on a compact Hausdorff space X are equivalently states on the C*-algebra $C(X)$. What about the other morphisms of $\mathbf{CSt}_{\mathrm{PU}}$? If we consider again the finite-dimensional case, a positive, unital map $\psi\colon \mathbb{C}^n \to \mathbb{C}^m$ defines a matrix

$$\begin{pmatrix} \psi(e_1) \\ \psi(e_2) \\ \vdots \\ \psi(e_n) \end{pmatrix} \tag{3.31}$$

where each column, from the first to the m^{th}, defines a distribution on $\{1,\dots,n\}$. As such ψ gives us a map $\{1,\dots,m\} \to \mathcal{R}(\{1,\dots,n\})$, or a Kleisli morphism $\{1,\dots,m\} \rightsquigarrow \{1,\dots,n\}$.

This gives an equivalence between the full subcategory of finite, discrete spaces of $\mathcal{K}l(\mathcal{R})$ and the full subcategory of $\mathbf{CSt}^{\mathrm{op}}_{\mathrm{PU}}$, or indeed $\mathbf{cCSt}^{\mathrm{op}}_{\mathrm{PU}}$, with objects $\mathbb{C}^n$ for $n \in \mathbb{N}$. It is a very pleasant result of Jacobs and Furber that extends this to a full categorical duality as a probabilistic analogue of the original Gelfand duality: the introduction of positive, unital maps into $\mathbf{cCSt}^{\mathrm{op}}_{\mathrm{MIU}}$ is exactly the introduction of probabilistic maps into $\mathbf{CH}$. This result was also proved in a different form by Parzygnat, with an explicit description that doesn't invoke the Riesz-Markov-Kakutani theorem [124].

Theorem 3.27 (Probabilistic Gelfand Duality [58, Thm. 5.1])**.** *There is an equivalence of categories* $\mathcal{K}l(\mathcal{R}) \cong \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{PU}}$ *extending the duality* $\mathbf{CH} \cong \mathbf{cCSt}_{\mathrm{MIU}}$ *of theorem 3.10.*

As such, we have the following diagram of categories:

$$\begin{tikzcd}
 & & \mathbf{CSt}^{\mathrm{op}}_{\mathrm{MIU}} \arrow[dll, tail] & & & \\
\mathbf{CSt}^{\mathrm{op}}_{\mathrm{PU}} \arrow[dd, two heads, "S"] & & & & \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{MIU}} \arrow[ull, two heads] \arrow[dll, tail] \arrow[ddd, equal, "\wr"] & \\
 & & \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{PU}} \arrow[ull, two heads] \arrow[d, equal, "\wr"] & & & \\
\mathbf{ConvCH} \arrow[d, equal, "\wr"] & & \mathcal{K}l(\mathcal{R}) \arrow[dll, two heads] \arrow[d, phantom, "\perp"] & & & \\
\mathcal{E}m(\mathcal{R}) \arrow[rrrr, "U_{\mathcal{E}m(\mathcal{R})}"'] & & & & \mathbf{CH} \arrow[ull, tail] \arrow[r, "U_{\mathbf{CH}}"'] & \mathbf{Set}
\end{tikzcd} \tag{3.32}$$

where double-headed arrows are fully faithful, and tailed arrows are wide.

3.2.4 C*-Tensor Products and Cross-Norms

Any formalism of quantum mechanics to be of any use modelling processes in the real world needs to deal with the combining of systems and the parallel composition of transformations. Given systems A and B, we need to have a new system within our formalism $A \otimes B$ which is result of preparing both systems A and B. Given states ρ_A, ρ_B on A and B respectively, we would like to be able to prepare a product state $\rho_A \otimes \rho_B$ on $A \otimes B$, which is a special case of that fact that given transformations $f_A \colon A \to A'$ and $f_B \colon B \to B'$, we should be able to construct a transformation $f_A \otimes f_B \colon A \otimes B \to A' \otimes B'$. In the operator algebraic approach to quantum foundations, as the notation above suggests, the role of this combination operator is taken by tensor products.

In the context of quantum systems represented with Hilbert spaces, we may use the Hilbert space structure to construct a tensor product. From Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we would like to take the tensor product of the C*-algebras $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$. An initial attempt might be simply to take the algebraic tensor product of these spaces $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. This has *-algebra operations taking multiplication and involution factor-wise via

$$(\phi \otimes \psi) \cdot (\phi' \otimes \psi') = \phi\phi' \otimes \psi\psi', \qquad (\phi \otimes \psi)^* = \phi^* \otimes \psi^*. \tag{3.33}$$

It is not yet a C*-algebra, since it is not equipped with a compatible norm.

An alternative direction uses the tensor product of the underlying Hilbert spaces. We may form their algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, and then complete with respect to the inner product

$$\left\langle \sum_j \alpha_j \otimes \beta_j, \sum_i \alpha_i \otimes \beta_i \right\rangle := \sum_{i,j} \langle \alpha_i, \alpha_j \rangle \langle \beta_i, \beta_j \rangle . \tag{3.34}$$

The resulting space is the *Hilbert space tensor product* and is notated $\mathcal{H}_1 \otimes_H \mathcal{H}_2$. It is the unique Hilbert space completion of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Consider the C*-algebra of bounded operators on this space, $\mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$. There are isometric injective *-homomorphisms $\mathcal{B}(\mathcal{H}_i) \hookrightarrow \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$ given by $\phi \mapsto \phi \otimes 1_{\mathcal{H}_2}$ and $\psi \mapsto 1_{\mathcal{H}_1} \otimes \psi$.

Since $\phi \otimes \psi = (\phi \otimes 1_{\mathcal{H}_2}) \cdot (1_{\mathcal{H}_1} \otimes \psi)$, there is an injective *-homomorphism $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \hookrightarrow \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$.

Given states $\rho_1 \colon \mathcal{B}(\mathcal{H}_1) \to \mathbb{C}$ and $\rho_2 \colon \mathcal{B}(\mathcal{H}_2) \to \mathbb{C}$, we may construct a product state $\rho_1 \otimes \rho_2 \colon \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2) \to \mathbb{C}$ with

$$\rho_1 \otimes \rho_2 \left(\sum_j \alpha_j \otimes \beta_j \right) = \sum_j \left(\rho_1 \left(\alpha_j \right) \rho_2 \left(\beta_j \right) \right) \tag{3.35}$$

which is a special case of the tensor product of maps which takes transformations $f_i \colon \mathcal{B}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H}'_i)$

$$\mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}'_1 \otimes_H \mathcal{H}'_2) \tag{3.36}$$

$$\sum_j \alpha_j \otimes \beta_j \mapsto \sum_j f_1 \left(\alpha_j \right) \otimes f_2 \left(\beta_j \right). \tag{3.37}$$

As such, $\mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$ is a C*-algebra in which we may represent the combination of the systems $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$. The inclusion $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \hookrightarrow \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$ bestows $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ with a norm with the property that, for all $\phi \in \mathcal{B}(\mathcal{H}_1)$ and $\psi \in \mathcal{B}(\mathcal{H}_2)$

$$\left\| \phi \otimes \psi \right\|_{\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)} = \left\| \phi \right\|_{\mathcal{B}(\mathcal{H}_1)} \left\| \psi \right\|_{\mathcal{B}(\mathcal{H}_2)}. \tag{3.38}$$

This norm has the C*-property, and as such the completion of $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ according to it gives a C*-algebra $\mathcal{B}(\mathcal{H}_1) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}_2) \subset \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$.

In fact, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional then the process above has $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1 \otimes_H \mathcal{H}_2)$ and any norm on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ which satisfies eq. (3.38), or equivalently (non-trivially) has the C*-property is equivalent to it. For a large class of C*-algebras, so called *nuclear* C*-algebras, a softer property continues to hold: if $\mathcal{A}$ is nuclear, for any other $\mathcal{B}$ there is only one way, up to isomorphism, to norm and complete the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ into a C*-algebra. However, this is hardly a general fact. Most pairs of C*-algebras admit non-equivalent norms with non-equivalent completions.

Definition 3.28 (Cross-norm)**.** *For two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ naturally has a multiplicative structure and an involution given by*

$$(a \otimes b) \cdot (a' \otimes b') := aa' \otimes bb', \qquad (a \otimes b)^* = a^* \otimes b^*. \tag{3.39}$$

A multiplication-contracting norm $\|\cdot\|_{\mathcal{A}\times\mathcal{B}}$ on $\mathcal{A} \otimes \mathcal{B}$ is called a cross-norm *if, for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$,*

$$\|a \otimes b\|_{\mathcal{A}\otimes\mathcal{B}} = \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}. \tag{3.40}$$

Given a cross-norm $\|\cdot\|$ on $\mathcal{A} \otimes \mathcal{B}$, the completion $\mathcal{A} \hat{\otimes} \mathcal{B}$ under this norm is a C*-algebra:

$$\begin{aligned} \|(a \otimes b)^*(a \otimes b)\| &= \|a^*a \otimes b^*b\| \\ &= \|a^*a\|_{\mathcal{A}} \|b^*b\|_{\mathcal{B}} \\ &= \|a\|^2_{\mathcal{A}} \|b\|^2_{\mathcal{B}} \\ &= \|a \otimes b\|^2. \end{aligned} \tag{3.41}$$

As such, cross-norms allow the construction of tensor products of C*-algebras.

The converse is also true: any C*-norm completion of $\mathcal{A} \otimes \mathcal{B}$ is also a cross-norm. Thus, our search for a norm on the algebraic tensor product which completes to a C*-algebra reduces to completions according to a cross-norm. As discussed above, in certain circumstances, this can be done uniquely, but in general, $\mathcal{A} \otimes \mathcal{B}$ may admit many inequivalent cross-norms.

We are saved somewhat from complete chaos by the existence of one point of order: for any pair of C*-algebras there exists a maximal cross-norm and a minimal cross-norm.

Proposition 3.29. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. There exist cross-norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ on $\mathcal{A} \otimes \mathcal{B}$ such that if $\|\cdot\|$ is a $\mathcal{A} \otimes \mathcal{B}$-cross-norm and $x \in \mathcal{A} \otimes \mathcal{B}$ then*

$$\|x\|_{\min} \leq \|x\| \leq \|x\|_{\max}. \tag{3.42}$$

The proof of this proposition is via the constructions of the next two sections: the tensor product introduced in section 3.2.5 is the minimal tensor product; the tensor product introduced in section 3.2.6 is the maximal one.

3.2.5 The Minimal Tensor Product

Section 3.2.4 motivated one way of understanding the tensor product of C*-algebras of the form $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. We may use this method for any pair of C*-algebras with the help of the Gelfand-Naimark theorem, theorem 3.11, which says that every C*-algebra $\mathcal{A}$ admits an isometric *-isomorphism with a closed subspace of $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ for some Hilbert space $\mathcal{H}_{\mathcal{A}}$.

As such, we may take our C*-algebras as having representations $\pi_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{A}})$ and $\pi_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{B}})$. Then

$$\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}} \colon \mathcal{A} \otimes \mathcal{B} \cong \pi_{\mathcal{A}}(\mathcal{A}) \otimes \pi_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}(\mathcal{H}_{\mathcal{B}}) \tag{3.43}$$

is an isometric *-homomorphism and

$$\mathcal{A} \otimes \mathcal{B} \cong \pi_{\mathcal{A}}(\mathcal{A}) \otimes \pi_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}_{\mathcal{B}}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes_H \mathcal{H}_{\mathcal{B}}) \tag{3.44}$$

bestows $\mathcal{A} \otimes \mathcal{B}$ with a cross-norm, under which we may complete.

Proposition 3.30 ([e.g. 63, Thm. 2])**.** *The cross-norm constructed above is independent of the choice of representation $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$.*

As such, it makes sense to speak of *the* cross-norm and completion according to this method:

Definition 3.31 (Minimal Tensor Product)**.** *Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, the unique cross-norm on $\mathcal{A} \otimes \mathcal{B}$ formed by taking representations $\pi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_{\mathcal{A}})$ and $\pi_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{B}(\mathcal{H}_{\mathcal{B}})$ and pulling back a norm via the representation $\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}} \colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes_H \mathcal{H}_{\mathcal{B}})$ is called* the spatial norm *or* the minimal (cross-)norm *and is notated by $\|\cdot\|_{\min}$. The completion of $\mathcal{A} \otimes \mathcal{B}$ by this norm is called the* minimal tensor product *and is notated by $\mathcal{A} \mathbin{\hat{\otimes}_{\min}} \mathcal{B}$.*

As the notation suggests, this is our minimal tensor product.

Proposition 3.32 ([e.g. 63, Thm. 4])**.** *Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, if $\|\cdot\|$ is a cross-norm on $\mathcal{A} \otimes \mathcal{B}$ then for all $x \in \mathcal{A} \otimes \mathcal{B}$,*

$$\|x\|_{\min} \leq \|x\|. \tag{3.45}$$

3.2.6 The Projective Tensor Product

The minimal C*-norm and thus the minimal tensor product were constructed explicitly via the inclusion

$$\mathcal{A} \otimes \mathcal{B} \cong \pi_{\mathcal{A}}(\mathcal{A}) \otimes \pi_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}(\mathcal{H}_{\mathcal{B}}) \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes_H \mathcal{H}_{\mathcal{B}}). \tag{3.46}$$

By Gelfand-Naimark, any C*-algebra completion $\mathcal{A} \hat{\otimes} \mathcal{B}$ of $\mathcal{A} \otimes \mathcal{B}$ must be similarly realised as an embedding

$$\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} \hat{\otimes} \mathcal{B} \hookrightarrow \mathcal{B}(\mathcal{H}) \tag{3.47}$$

for some Hilbert space $\mathcal{H}$. For the maximal norm, it is sufficient to show that all cross-norms are bounded above, and then to take the supremum over all of them to define $\|\cdot\|_{\max}$.

Lemma 3.33. *Let $\|\cdot\|$ be a cross-norm on $\mathcal{A} \otimes \mathcal{B}$ for C*-algebras $\mathcal{A}, \mathcal{B}$. Then for $x \in \mathcal{A} \otimes \mathcal{B}$*

$$\|x\| \leq \inf \left\{ \sum_{i=1}^{n} \|a_i\|_{\mathcal{A}} \|b_i\|_{\mathcal{B}} \,\middle|\, x = \sum_{i=1}^{n} a_i \otimes b_i, a_i \in \mathcal{A}, b_i \in \mathcal{B} \right\}. \tag{3.48}$$

Proof. Let $\|\cdot\|$ be such a cross-norm and $x = \sum_{i=1}^{n} a_i \otimes b_i$ a decomposition of $x \in \mathcal{A} \otimes \mathcal{B}$. Then $\|x\| = \|\sum_{i=1}^{n} a_i \otimes b_i\| \leq \sum_{i=1}^{n} \|a_i \otimes b_i\| = \sum_{i=1}^{n} \|a_i\| \|b_i\|$. □

Definition 3.34 (Projective Tensor Product)**.** *Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, the* maximal (cross-)norm *is given by*

$$\|x\|_{\max} := \sup \left\{ \|x\|_{\mathcal{A}\otimes\mathcal{B}} \,\middle|\, \|\cdot\|_{\mathcal{A}\otimes\mathcal{B}} \text{ is a cross-norm on } \mathcal{A} \otimes \mathcal{B} \right\} \tag{3.49}$$

The completion of $\mathcal{A} \otimes \mathcal{B}$ by this norm is called the projective *or* maximal tensor product *and is notated by $\mathcal{A} \hat{\otimes}_{\max} \mathcal{B}$.*

As a necessary sanity check, we show that the maximal norm is indeed a cross-norm.

Proposition 3.35. *$\|\cdot\|_{\max}$ is a cross-norm.*

Proof. Positive definiteness and homogeneity are straightforward. For the triangle inequality, note that for any $x, y \in \mathcal{A} \otimes \mathcal{B}$ and $\varepsilon > 0$, there exists a cross-norm $\|\cdot\|$ such that

$$\begin{aligned} \|x+y\|_{\max} &\leq \|x+y\| + \varepsilon \\ &\leq \|x\| + \|y\| + \varepsilon \\ &\leq \|x\|_{\max} + \|y\|_{\max} + \varepsilon. \end{aligned} \tag{3.50}$$

Since ε is arbitrary, $\|x+y\|_{\max} \leq \|x\|_{\max} + \|y\|_{\max}$. It is a cross-norm, since for $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$

$$\begin{aligned} \|a \otimes b\|_{\max} &= \sup\left\{\|a \otimes b\| \mid \|\cdot\| \text{ is a cross-norm on } \mathcal{A} \otimes \mathcal{B}\right\} \\ &= \sup\left\{\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}} \mid \|\cdot\| \text{ is a cross-norm on } \mathcal{A} \otimes \mathcal{B}\right\} \\ &= \|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}. \end{aligned} \tag{3.51}$$

□

By the definition of $\|\cdot\|_{\max}$ as the supremum of cross-norms, we have that it is the maximal cross-norm.

Corollary 3.36. *Let $\|\cdot\|$ be a cross-norm on $\mathcal{A} \otimes \mathcal{B}$, then for any $x \in \mathcal{A} \otimes \mathcal{B}$*

$$\|x\| \leq \|x\|_{\max}. \tag{3.52}$$

Together with proposition 3.32, this affirms proposition 3.29, that all cross-norms fall between $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$.

The following proposition is from Takesaki and will be of use later.

Proposition 3.37 ([152, p189])**.** *The maximal cross-norm on $\mathcal{A} \otimes \mathcal{B}$ may be equivalently defined as*

$$\|x\|_{\max} = \inf\left\{\sum_i \|a_i\|\|b_i\| \,\middle|\, x = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}\right\}. \tag{3.53}$$

3.2.7 Nuclear C*-algebras

With at least two tensor products on C*-algebras, we can ask if and when they are distinct. We noted in section 3.2.4 that for finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there is only one cross-norm on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. In this section, we briefly recount some theorems about which algebras have this property.

Definition 3.38 (Nuclear C*-algebra)**.** *A C*-algebra $\mathcal{A}$ is called* nuclear *if for every C*-algebra $\mathcal{B}$, $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $\mathcal{A} \otimes \mathcal{B}$.*

In such a case, we may say that $\mathcal{A} \hat{\otimes}_{\min} \mathcal{B} = \mathcal{A} \hat{\otimes}_{\max} \mathcal{B}$ with common dense subset $\mathcal{A} \otimes \mathcal{B}$. As such we may speak of the cross-norm $\|\cdot\|_{\mathcal{A}\otimes\mathcal{B}}$ or the tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$.

The two important classes of nuclear C*-algebras are finite-dimensional C*-algebras and commutative C*-algebras.

Proposition 3.39 (Props. C.100, C.101)**.** *If a C*-algebra is finite-dimensional or commutative, it is nuclear.*

In fact, in both cases, the tensor product has a nice form: if $\mathcal{A}$ is finite-dimensional then for any $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ is already complete under the cross-norm, so $\mathcal{A} \hat{\otimes} \mathcal{B} \cong (\mathcal{A} \otimes \mathcal{B}, \|\cdot\|)$.

For $X \in \mathbf{CH}$ and $\mathcal{B} \in \mathbf{CSt}_{\mathrm{PU}}$ considered as a topological space, $C(X) \hat{\otimes} \mathcal{B} \cong C(X; \mathcal{B}) = \{f \colon X \to \mathcal{B} \mid f \text{ continuous}\}$ with the supremum norm.

Corollary 3.40. *Let $\mathcal{H}$ be a finite-dimensional Hilbert Space. Then $\mathcal{B}(\mathcal{H})$ is nuclear.*

Proof. If $\mathcal{H}$ has finite dimension then $\dim(\mathcal{B}(\mathcal{H})) = \dim(\mathcal{H})^2$, hence $\mathcal{B}(\mathcal{H})$ is nuclear. ☐

3.3 Complete Positivity and Monoidal Structures for C*-algebras

In order to incorporate these ideas into the category theoretic framework, we have to consider how the various tensor products on algebras extend to morphisms.

Ideally, we'd want the tensor products of definition 3.31 and definition 3.34 to define monoidal structures on our various categories of C*-algebras.

For any C*-algebra $\mathcal{A}$, the algebraic tensor product $\mathcal{A} \otimes \mathbb{C}$ is isomorphic to $\mathcal{A}$ under the isomorphism $a \otimes \lambda \mapsto \lambda a$ so $\mathcal{A} \mathbin{\hat{\otimes}_{\mathrm{min}}} \mathbb{C} \cong \mathcal{A} \cong \mathcal{A} \mathbin{\hat{\otimes}_{\mathrm{max}}} \mathbb{C}$. As such $\mathbb{C}$ is a good candidate for the monoidal unit.

Proposition 3.41. *Let $f \colon \mathcal{A} \to \mathcal{A}'$ and $g \colon \mathcal{B} \to \mathcal{B}'$ be bounded linear maps between C*-algebras and let $\mathcal{A} \mathbin{\hat{\otimes}_\alpha} \mathcal{B}$ and $\mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}'$ be C*-algebra completions of $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{A}' \otimes \mathcal{B}'$ respectively. Then there exist unique bounded linear maps*

$$f \;{}_{\mathrm{max}}\hat{\otimes}_\beta\; g \colon \mathcal{A} \mathbin{\hat{\otimes}_{\mathrm{max}}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}' \tag{3.54}$$

and

$$f \;{}_{\alpha}\hat{\otimes}_{\mathrm{min}}\; g \colon \mathcal{A} \mathbin{\hat{\otimes}_\alpha} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\mathrm{min}}} \mathcal{B}' \tag{3.55}$$

such that the following diagram commutes in $\mathbf{Vect}_{\mathbb{C}}$

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}_{\mathrm{max}}} \mathcal{B} \arrow[r, "f_{\mathrm{max}}\hat{\otimes}_\beta g"] & \mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}' \\
\mathcal{A} \otimes \mathcal{B} \arrow[u, hook] \arrow[d, hook] \arrow[r, "f \otimes g"] & \mathcal{A}' \otimes \mathcal{B}' \arrow[u, hook] \arrow[d, hook] \\
\mathcal{A} \mathbin{\hat{\otimes}_\alpha} \mathcal{B} \arrow[r, "f_\alpha\hat{\otimes}_{\mathrm{min}} g"'] & \mathcal{A}' \mathbin{\hat{\otimes}_{\mathrm{min}}} \mathcal{B}'
\end{tikzcd} \tag{3.56}$$

*Additionally, if f and g are *-homomorphisms, so are $f \;{}_{\mathrm{max}}\hat{\otimes}_\beta\; g$ and $f \;{}_{\alpha}\hat{\otimes}_{\mathrm{min}}\; g$.*

The following lemma will be useful in proving this theorem and for further reasoning about extending maps on the algebraic tensor product:

Lemma 3.42. *Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be Banach spaces. Let $C \subset A$ be a linear subspace. If $\phi \colon C \to B$ is a bounded linear map, then it extends uniquely to a bounded linear map $\hat{\phi} \colon \overline{C} \to B$, where $\overline{C}$ is the topological closure of C in A.*

*If in addition these spaces are *-algebras then if ϕ preserves the unit, involution, or multiplication, so does $\hat{\phi}$.*

Proof of lemma 3.42. The existence of such a unique continuous extension is well explored.

Preservation of the unit is trivial, since ϕ can only be unital if $1 \in C$ and the restriction of $\hat{\phi}$ to C is ϕ.

Preservation of multiplication and involution follows from the continuity of these operations: take $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ sequences in C converging to a and $b \in \overline{C}$ respectively. Then

$$\hat{\phi}\left(a^*\right) = \lim_n \left[\phi(a_n^*)\right] = \lim_n \left[\phi(a_n)^*\right] = \left(\lim_n \left[\phi(a_n)\right]\right)^* = \left(\hat{\phi}(a)\right)^* \tag{3.57}$$

and

$$\hat{\phi}(ab) = \hat{\phi}(\lim_n a_n b_n) = \lim_n \phi(a_n b_n) = \lim_n \phi(a_n) \cdot \phi(b_n) = \lim_n \phi(a_n) \cdot \lim_n \phi(b_n) \tag{3.58}$$

which is $\hat{\phi}(a_n) \cdot \hat{\phi}(b_n)$ as desired. □

Proof of proposition 3.41. The map $f \otimes g$ exists by the universal property of the tensor product: the map $(a, b) \mapsto f(a) \otimes g(b)$ is bilinear, since f and g are both linear. If we are working with *-homomorphisms, $f \otimes g$ is multiplicative, unital and preserves the involution.

We may compose with the inclusions $\mathcal{A}' \otimes \mathcal{B}' \hookrightarrow \mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}'$ or $\mathcal{A}' \otimes \mathcal{B}' \hookrightarrow \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}'$, so it only remains to show that composite maps $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}'$ and $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}'$ factor through the dense inclusions $\mathcal{A} \otimes \mathcal{B} \hookrightarrow \mathcal{A} \mathbin{\hat{\otimes}_{\max}} \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{B} \hookrightarrow \mathcal{A} \mathbin{\hat{\otimes}_\alpha} \mathcal{B}$ respectively. We approach the former first.

Claim. *The map* $f \otimes g \colon (\mathcal{A} \otimes \mathcal{B}, \|\cdot\|_{\max}) \to \mathcal{A}' \mathbin{\hat{\otimes}_\beta} \mathcal{B}'$ *is bounded.*

Let $x \in \mathcal{A} \otimes \mathcal{B}$ and let $\varepsilon > 0$. We may choose a decomposition $x = \sum_{i=1}^n a_i \otimes b_i$ of x by proposition 3.37 such that $\sum_{i=1}^n \|a_i\| \|b_i\| \leq \|x\|_{\max} + \varepsilon$.

$$\begin{aligned}
\frac{\|f\otimes g\,(x)\|}{\|x\|_{\max}} &= \frac{\|f\otimes g\,(\sum_i a_i\otimes b_i)\|}{\|x\|_{\max}} \\
&= \frac{\|\sum_i f(a_i)\otimes g(b_i)\|}{\|x\|_{\max}} \\
&\leq \frac{\sum_i \|f(a_i)\otimes g(b_i)\|}{\|x\|_{\max}} \\
&= \frac{\sum_i \|f(a_i)\|\,\|g(b_i)\|}{\|x\|_{\max}} \\
&\leq \|f\|\|g\|\frac{\sum_i \|a_i\|\|b_i\|}{\|x\|_{\max}} \\
&\leq \|f\|\|g\|\frac{\|x\|_{\max}+\varepsilon}{\|x\|_{\max}} \\
&\leq \|f\|\|g\|\left(1+\frac{\varepsilon}{\|x\|_{\max}}\right).
\end{aligned} \tag{3.59}$$

Thus, $\|f\otimes g\| \leq \|f\|\|g\|$.

By lemma 3.42, it extends uniquely to $f \,{}_{\max}\hat{\otimes}_{\beta}\, g \colon \mathcal{A}\,\hat{\otimes}_{\max}\,\mathcal{B} \to \mathcal{A}'\,\hat{\otimes}_{\beta}\,\mathcal{B}'$, and is a *-homomorphism if f and g are.

Now we approach the extension of $\mathcal{A}\otimes\mathcal{B} \to \mathcal{A}'\,\hat{\otimes}_{\min}\,\mathcal{B}'$.

Claim. *The map* $f\otimes g\colon (\mathcal{A}\otimes\mathcal{B}, \|\cdot\|_{\alpha}) \to \mathcal{A}'\,\hat{\otimes}_{\min}\,\mathcal{B}'$ *is bounded.*

Again, let $x\in\mathcal{A}\otimes\mathcal{B}$. Let $\pi_{\mathcal{A}'}\colon \mathcal{A}'\to\mathcal{B}(\mathcal{H}_{\mathcal{A}'})$ and $\pi_{\mathcal{B}'}\colon \mathcal{B}'\to\mathcal{B}(\mathcal{H}_{\mathcal{B}'})$ be faithful representations of $\mathcal{A}'$ and $\mathcal{B}'$ respectively.

Then

$$\begin{aligned}
\frac{\|f\otimes g(x)\|_{\min}}{\|x\|_{\alpha}} &= \frac{\|(\pi_{\mathcal{A}'}\otimes\pi_{\mathcal{B}'})\circ(f\otimes g)(x)\|}{\|x\|_{\alpha}} \\
&= \frac{\|(\pi_{\mathcal{A}'}\circ f)\otimes(\pi_{\mathcal{B}'}\circ g)(x)\|}{\|x\|_{\alpha}} \\
&\leq \frac{\|x\|_{\min}}{\|x\|_{\alpha}} \leq 1.
\end{aligned} \tag{3.60}$$

Thus, $\|f\otimes g\| \leq 1$.

Another application of lemma 3.42 gives us $f \,{}_{\alpha}\hat{\otimes}_{\min}\, g$ and shows it is a *-homomorphism. □

This proposition has a number of important corollaries.

Corollary 3.43. *Given* $f\colon \mathcal{A} \to \mathcal{A}'$ *and* $g\colon \mathcal{B} \to \mathcal{B}'$ *in* $\mathbf{CSt}_{\mathrm{MIU}}$*, there exist unique *-homomorphisms*

$$f \mathbin{\hat{\otimes}_{\min}} g\colon \mathcal{A} \mathbin{\hat{\otimes}_{\min}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}' \tag{3.61}$$

and

$$f \mathbin{\hat{\otimes}_{\max}} g\colon \mathcal{A} \mathbin{\hat{\otimes}_{\max}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\max}} \mathcal{B}' \tag{3.62}$$

such that the below diagram commutes in $\mathbf{Vect}_{\mathbb{C}}$

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}_{\max}} \mathcal{B} \arrow[r, "f \hat{\otimes}_{\max} g"] & \mathcal{A}' \mathbin{\hat{\otimes}_{\max}} \mathcal{B}' \\
\mathcal{A} \otimes \mathcal{B}, \arrow[r, "f \otimes g"] \arrow[u, hook] \arrow[d, hook] & \mathcal{A}' \otimes \mathcal{B}' \arrow[u, hook] \arrow[d, hook] \\
\mathcal{A} \mathbin{\hat{\otimes}_{\min}} \mathcal{B} \arrow[r, "f \hat{\otimes}_{\min} g"'] & \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}'
\end{tikzcd} \tag{3.63}$$

Theorem 3.44. *The triples* $\left(\mathbf{CSt}_{\mathrm{MIU}}, \hat{\otimes}_{\min}, \mathbb{C}\right)$ *and* $\left(\mathbf{CSt}_{\mathrm{MIU}}, \hat{\otimes}_{\max}, \mathbb{C}\right)$ *are symmetric monoidal categories.*

For the maximal tensor product, see Landsman [114, p. 773], who cites Hofmann [76]. This is a property that descends from structure on Banach spaces. The infimum definition of the maximal cross-norm of proposition 3.37 may be defined for any Banach space, and this *projective tensor product of Banach spaces* makes **Ban**, the category of Banach spaces and short maps, linear maps of norm less than or equal to one, into a symmetric monoidal category. All *-homomorphisms are short, and the associator, unitors and symmetry in **Ban** lift to *-homomorphisms:

$$\left(a \mathbin{\hat{\otimes}_{\max}} \left(b \mathbin{\hat{\otimes}_{\max}} c\right)\right)^* = a^* \mathbin{\hat{\otimes}_{\max}} \left(b \mathbin{\hat{\otimes}_{\max}} c\right)^* = a^* \mathbin{\hat{\otimes}_{\max}} \left(b^* \mathbin{\hat{\otimes}_{\max}} c^*\right) \tag{3.64}$$

whose image under the associator will be

$$\left(a^* \mathbin{\hat{\otimes}_{\max}} b^*\right) \mathbin{\hat{\otimes}_{\max}} c^* = \left(\left(a \mathbin{\hat{\otimes}_{\max}} b\right) \mathbin{\hat{\otimes}_{\max}} c\right)^*, \tag{3.65}$$

for example.

Thus, there is a forgetful functor $\mathbf{CSt}_{\mathrm{MIU}} \to \mathbf{Ban}$ which picks up the coherence of $\hat{\otimes}_{\max}$. Proposition 3.41 tells us that the monoidal product of morphisms translates too.

The fact that the projective tensor product on **Ban** gives a symmetric monoidal category seems to be folklore. We could not find a direct proof, but it is stated and used regularly [e.g. 42, p. 2, 119, Chap. 3, 122, 145, Sec. 4.4]. Proofs do exist of the existence of unit and associativity isomorphisms extending those of the algebraic tensor product, which is a monoidal product on the underlying vector spaces [e.g. 24, Thm. 2.17, 26, § II.1.8]. Commutativity of the relevant coherence diagrams follow from those of algebraic tensor product and noting that the algebraic tensor product of the underlying Banach spaces is dense in the projective tensor product of those Banach spaces.

For the minimal tensor product, this is again hard to find a direct source for and appears again to be folklore, this time in the area of K-theory. See, for example, Meyer [120, p. 21] or Bunke [20, Prop. 11.31], who proves this for C*-categories (C*-algebras being then single objects C*-categories), and who both also treat the maximal tensor product.

Since commutative C*-algebras are nuclear, these agree on $\mathbf{cCSt}_{\mathrm{MIU}}$, so we will refer to *the* monoidal category $\left(\mathbf{CSt}_{\mathrm{MIU}}, \hat{\otimes}, \mathbb{C}\right)$.

Additionally, we may consider the relationship between the Cartesian monoidal structure on **CH** and the monoidal structure of $\mathbf{cCSt}_{\mathrm{MIU}}$.

Proposition 3.45 (p. 773). *The functor* $C\colon \mathbf{CH} \to \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{MIU}}$ *is a strong monoidal equivalence. Explicitly,* C *is an equivalence of categories and there are isomorphisms* $C(\{*\}) \cong \mathbb{C}$ *and*

$$C(X \times Y) \cong C(X) \mathbin{\hat{\otimes}} C(Y) \tag{3.66}$$

natural in X *and* Y*.*

The base equivalence of categories is the already-stated Gelfand duality, theorem 3.10, now with the additional monoidal structure. The latter isomorphisms take $f \otimes g \in C(X) \mathbin{\hat{\otimes}} C(Y)$, for $f \in C(X)$ and $g \in C(Y)$, to the function $fg \in C(X \times Y)$ with $fg(x, y) = f(x)g(y)$.

It makes sense to ask whether the monoidal structures on $\mathbf{cCSt}_{\mathrm{MIU}}$ and $\mathbf{CSt}_{\mathrm{MIU}}$ can also be extended along the inclusions into $\mathbf{cCSt}_{\mathrm{PU}}$ and $\mathbf{CSt}_{\mathrm{PU}}$.

We may tensor positive, unital maps in the same way as in proposition 3.41, but positivity is not preserved automatically.

Proposition 3.46. *Let $\phi\colon \mathcal{A} \to \mathcal{A}'$ and $\psi\colon \mathcal{B} \to \mathcal{B}'$ be positive, unital maps between C*-algebras. Then the unique bounded linear maps*

$$f \mathbin{\hat{\otimes}_{\min}} g\colon \mathcal{A} \mathbin{\hat{\otimes}_{\min}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}' \tag{3.67}$$

and

$$f \mathbin{\hat{\otimes}_{\max}} g\colon \mathcal{A} \mathbin{\hat{\otimes}_{\max}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\max}} \mathcal{B}' \tag{3.68}$$

are unital, but may not be positive.

Proof. Proposition 3.41 gives the bounded maps and that they are unital.

The archetypical example of a positive maps which does not remain so after taking its tensor with another positive map, as given in Paulsen [126] and elsewhere, is the transpose map and the identity map on complex matrices. For $n \in \mathbb{N}$, transposition is the bounded linear map

$$\begin{aligned} (\cdot)^\intercal\colon M^n(\mathbb{C}) &\to M^n(\mathbb{C}) \\ (a_{ij})_{i,j} &\mapsto (a_{ji})_{i,j}\,. \end{aligned} \tag{3.69}$$

Transposition is positive, since the transpose of a positive semi-definite matrix is positive semi-definite.

For the algebraic tensor product we have an isomorphism $\mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2} \cong \mathbb{C}^{4\times 4}$ given by

$$A \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + C \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + D \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.70}$$

for matrix blocks $A, B, C, D \in \mathbb{C}^{2\times 2}$. It has only one C*-norm, and is already complete with respect to it.

The map $(\cdot)^\intercal \otimes \mathrm{id}\colon \mathbb{C}^{4\times 4} \to \mathbb{C}^{4\times 4}$ takes

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A^\intercal & B^\intercal \\ C^\intercal & D^\intercal \end{pmatrix} \tag{3.71}$$

and as such

$$(\cdot)^\intercal \otimes \mathrm{id} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.72}$$

The former matrix is positive, but the latter certainly is not. □

Definition 3.47 (Complete Positivity)**.** *Given two *-algebras $\mathcal{A}, \mathcal{B}$, a linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is called* completely positive *if for all $n \in \mathbb{N}_0$*

$$\phi \otimes \mathrm{id}_{M^n(\mathbb{C})} \colon \mathcal{A} \otimes M^n(\mathbb{C}) \to \mathcal{B} \otimes M^n(\mathbb{C}) \tag{3.73}$$

is positive. In particular, a completely positive map is positive.

Note that $\mathcal{A} \otimes M^n(\mathbb{C})$ and $\mathcal{B} \otimes M^n(\mathbb{C})$ do not need completing under their unique C*-norm, since $M^n(\mathbb{C})$ is finite dimensional.

Definition 3.48. *Given a C*-algebra $\mathcal{A}$, we write $M^n(\mathcal{A})$ for the set of $n \times n$ matrices with entries in $\mathcal{A}$. An element of $M^n(\mathcal{A})$ maybe denoted by $A = (a_{i,j})$, for $a_{i,j} \in \mathcal{A}$. With matrix multiplication and involutive transposition, $(a_{i,j})^* = (a^*_{j,i})$, $M^n(\mathcal{A})$ is a *-algebra.*

In fact, $\mathcal{A} \otimes M^n(\mathbb{C}) \cong M^n(\mathcal{A})$ as *-algebras, and the latter inherits the unique norm of the former. Given a bounded linear map $\phi\colon \mathcal{A} \to \mathcal{B}$, $\phi \otimes \mathrm{id}_{M^n(\mathbb{C})}$ corresponds to the map

$$\begin{aligned} \phi^{(n)} \colon M^n(\mathcal{A}) &\to M^n(\mathcal{B}) \\ (a_{ij}) &\mapsto (\phi(a_{ij})) . \end{aligned} \tag{3.74}$$

Thus, $\phi\colon \mathcal{A} \to \mathcal{B}$ is completely positive if and only if $\phi^{(n)}\colon M^n(\mathcal{A}) \to M^n(\mathcal{B})$ is positive for all $n \in \mathbb{N}_0$.

Proposition 3.49 ([18, Thm. 3.5.3])**.** *If $\phi\colon \mathcal{A} \to \mathcal{B}$ and $\psi\colon \mathcal{A}' \to \mathcal{B}'$ are completely positive, then the unique bounded linear maps $\phi \mathbin{\hat{\otimes}_{\min}} \psi\colon \mathcal{A} \mathbin{\hat{\otimes}_{\min}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\min}} \mathcal{B}'$ and $\phi \mathbin{\hat{\otimes}_{\max}} \psi\colon \mathcal{A} \mathbin{\hat{\otimes}_{\max}} \mathcal{B} \to \mathcal{A}' \mathbin{\hat{\otimes}_{\max}} \mathcal{B}'$ are completely positive, and in particular, are positive.*

Definition 3.50 (Category of C*-algebras and Completely Positive Maps)**.** *The* category of C*-algebras and completely positive maps *is the wide subcategory of* $\mathbf{CSt}_{\mathrm{PU}}$ *of all C*-algebras and those positive, unital maps that are also completely positive.*

Proposition 3.51 ([126, Thm. 3.9 and Thm. 3.11])**.** *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a positive, unital map. If either of $\mathcal{A}$ or $\mathcal{B}$ are commutative, then ϕ is completely positive.*

This gives rise to two important categorical corollaries:

Corollary 3.52. $\mathbf{cCSt}_{\mathrm{PU}}$ *is a full subcategory of* $\mathbf{CSt}_{\mathrm{CPU}}$*,*

We may add this information to diag. (3.22)

$$\begin{tikzcd}
 & \mathbf{cCSt}_{\mathrm{PU}} \arrow[d, two heads] \arrow[dr, two heads] & \\
\mathbf{cCSt}_{\mathrm{MIU}} \arrow[ur, tail] \arrow[r] \arrow[dr, two heads] & \mathbf{CSt}_{\mathrm{CPU}} \arrow[r, tail] & \mathbf{CSt}_{\mathrm{PU}} \\
 & \mathbf{CSt}_{\mathrm{MIU}} \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd} \tag{3.75}$$

where double-headed arrows are fully faithful, and tailed arrows are essentially surjective (in this case, wide).

Corollary 3.53. *Any state* $\rho\colon \mathcal{A} \to \mathbb{C}$ *on a* C^**-algebra* $\mathcal{A}$ *is completely positive, which is to say that* $\mathbf{CSt}_{\mathrm{CPU}}(\mathcal{A}, \mathbb{C}) = \mathbf{CSt}_{\mathrm{PU}}(\mathcal{A}, \mathbb{C})$*.*

We may add to *this* information to diag. (3.26)

$$\begin{tikzcd}
\mathbf{CSt}_{\mathrm{CPU}}^{\mathrm{op}} \arrow[r, tail] \arrow[rrrr, bend left, "\mathbf{CSt}_{\mathrm{CPU}}(-{,}\mathbb{C})"] & \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \arrow[r, two heads, "S"'] \arrow[rrr, bend right, "\mathbf{CSt}_{\mathrm{PU}}(-{,}\mathbb{C})"'] & \mathbf{ConvCH} \arrow[r, "U_{\mathbf{ConvCH}}"'] & \mathbf{CH} \arrow[r, "U_{\mathbf{CH}}"'] & \mathbf{Set}
\end{tikzcd} \tag{3.76}$$

Additionally, we have that $\mathbf{cCSt}_{\mathrm{CPU}} = \mathbf{cCSt}_{\mathrm{PU}}$.

Given proposition 3.49, the monoidal structures on $\mathbf{CSt}_{\mathrm{MIU}}$ generalise along the inclusion $\mathbf{CSt}_{\mathrm{MIU}} \to \mathbf{CSt}_{\mathrm{CPU}}$.

Theorem 3.54 (Monoidal Structures on $\mathbf{CSt}_{\mathrm{CPU}}$)**.** *The triples* $\left(\mathbf{CSt}_{\mathrm{CPU}}, \hat{\otimes}_{\min}, \mathbb{C}\right)$ *and* $\left(\mathbf{CSt}_{\mathrm{CPU}}, \hat{\otimes}_{\max}, \mathbb{C}\right)$ *are symmetric monoidal categories.*

The category $\mathbf{CSt}_{\mathrm{CPU}}$ is the natural place to combine quantum mechanics and classical probability. For the latter, it contains the full subcategory, $\mathbf{cCSt}_{\mathrm{PU}}$, which is equivalent to $\mathcal{K}l(\mathcal{R})^{\mathrm{op}}$. For the former, the quantum channels $\mathcal{H}_1 \to \mathcal{H}_2$ are exactly the elements of $\mathbf{CSt}_{\mathrm{CPU}}\left(\mathcal{B}(\mathcal{H}_2), \mathcal{B}(\mathcal{H}_1)\right)$. Objects in this category can represent situations with both classical and quantum aspects, and the morphisms can encode the transformation of these circumstances via both classical and quantum processes. The monoidal structure allows the construction of composite systems. We will find that this is exactly where we would like to situate our symmetry theorems.

3.4 Quantum de Finetti Theorems

We are now equipped with the tools to revisit the content of section 3.1. In this section, we state the two quantum de Finetti theorems that are treated as limits in section 3.5. In getting to that point, it is necessary to define exchangeability in the context of the tensor product of quantum states, and to do that we need the infinite tensor power of a C*-algebra.

3.4.1 Product and Symmetric States

Proposition 3.55 (C. 98). *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and let $\mathcal{A} \hat{\otimes} \mathcal{B}$ be some C*-norm completion tensor product. Let $\rho_{\mathcal{A}} \in S(\mathcal{A})$ and $\rho_{\mathcal{B}} \in S(\mathcal{B})$ be states. Then there is a unique state $\rho_{\mathcal{A}} \hat{\otimes} \rho_{\mathcal{B}} \in S\big(\mathcal{A} \hat{\otimes} \mathcal{B}\big)$, called* the product state*, such that for $\sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$:*

$$\rho_{\mathcal{A}} \hat{\otimes} \rho_{\mathcal{B}} \left(\sum_i a_i \otimes b_i \right) = \sum_i \left(\rho_{\mathcal{A}}(a_i)\, \rho_{\mathcal{B}}(b_i) \right). \tag{3.77}$$

Definition 3.56 (Tensor Powers of C*-algebras and States). *Given a C*-algebra $\mathcal{A}$,* the n^{th} tensor power of $\mathcal{A}$ *is*

$$\mathcal{A}^{\otimes n} := \underbrace{\mathcal{A} \hat{\otimes} \ldots \hat{\otimes} \mathcal{A}}_{n \text{ times}}. \tag{3.78}$$

In the monoidal categories, $\left(\mathbf{CSt}_{\text{CPU}}, \hat{\otimes}_{\min}, \mathbb{C}\right)$ and $\left(\mathbf{CSt}_{\text{CPU}}, \hat{\otimes}_{\max}, \mathbb{C}\right)$, this definition is precise. If $\mathcal{A}$ is not nuclear, these powers may differ under each monoidal product. In the case that the choice of monoidal structure is not clear we will write

$$\mathcal{A}^{\otimes n}_{\min} := \underbrace{\mathcal{A} \hat{\otimes}_{\min} \ldots \hat{\otimes}_{\min} \mathcal{A}}_{n \text{ times}} \quad \text{and} \quad \mathcal{A}^{\otimes n}_{\max} := \underbrace{\mathcal{A} \hat{\otimes}_{\max} \ldots \hat{\otimes}_{\max} \mathcal{A}}_{n \text{ times}}. \tag{3.79}$$

Given $\mathcal{A}^{\otimes n}$ *and* $\rho \in S(\mathcal{A})$*, we then may form* the n^{th} product power of ρ, *iterating proposition 3.55, as the unique state* $\rho^{\otimes n} \in S(\mathcal{A}^{\otimes n})$ *with the property that*

$$\rho^{\otimes n}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \rho(a_1)\rho(a_2)\cdots\rho(a_n) \tag{3.80}$$

for $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n \text{ times}} \subset \mathcal{A}^{\otimes n}$*. This is the composition under the monoidal structure*

$$\mathcal{A}^{\otimes n} \xrightarrow{\rho\otimes\cdots\otimes\rho} \mathbb{C}^{\otimes n} \xrightarrow{\sim} \mathbb{C} \tag{3.81}$$

Definition 3.57 (Symmetric State)**.** *For a C*-algebra* $\mathcal{A}$*, the symmetric group on* n *elements* $\mathcal{S}_n$ *defines an action on* $\mathcal{A}^{\otimes n}$ *by permuting the factors of the tensor product. For a permutation* $\sigma \in \mathcal{S}_n$*, we write*

$$\begin{aligned} \eta_\sigma \colon \mathcal{A}^{\otimes n} &\to \mathcal{A}^{\otimes n} \\ \eta_\sigma\left(a_1 \otimes \cdots \otimes a_n\right) &:= a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}, \end{aligned} \tag{3.82}$$

extending via the inclusion $\underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n \text{ times}} \subset \mathcal{A}^{\otimes n}$.

These maps are closely related to the braiding maps of the monoidal categories of C-algebras from definition 2.11. Indeed, we have* $\eta_\sigma = \mathcal{A}^{\otimes(\sigma^{-1})} = (\mathcal{A}^{\otimes\sigma})^{-1}$*. When we take the opposites of these categories for topological results, they are exactly the monoidal braiding maps. They are induced by repeated use of the monoidal symmetry* $\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \xrightarrow{\sim} \mathcal{B} \mathbin{\hat{\otimes}} \mathcal{A}$*.*

The action also induces an action on $S(\mathcal{A}^{\otimes n})$ *by pre-composition with* η_σ*. A state* $\rho \in S(\mathcal{A}^{\otimes n})$ *is* symmetric *if it is invariant under this action. Explicitly,* ρ *is symmetric if, for every permutation* $\sigma \in \mathcal{S}_n$ *and every* $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n \text{ times}}$*,*

$$\rho\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = \rho\left(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}\right), \tag{3.83}$$

since the span of such elements is dense in $\mathcal{A}^{\otimes n}$*.*

Proposition 3.58. *For a given state* $\rho \in S(\mathcal{A})$*, the product state* $\rho^{\otimes n} \in S(\mathcal{A}^{\otimes n})$ *is symmetric.*

Proof. This, of course, is just because

$$\begin{aligned}\rho^{\otimes n}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) &= \rho(a_1)\rho(a_2)\cdots\rho(a_n) \\ &= \rho(a_{\sigma^{-1}(1)})\rho(a_{\sigma^{-1}(2)})\cdots\rho(a_{\sigma^{-1}(n)}) \\ &= \rho\left(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}\right).\end{aligned} \tag{3.84}$$

□

3.4.2 Infinite Tensor Products, States and Exchangeability

Tensor powers $\mathcal{A}^{\otimes n}$ of C*-algebras model repeated systems. Tensor product states $\rho^{\otimes n}$ model unentangled preparations of the same experiments over and over. To state the quantum de Finetti theorems we will need (countably) infinite tensor powers $\mathcal{A}^{\otimes \mathbb{N}}$ and infinite product states $\rho^{\otimes \mathbb{N}}$ to model infinite lists, or streams, of apparatuses and results, similar to modelling an endless coin flipping experiment, which allows us to talk about tail behaviour of such a stream.

The source for the definitions of infinite tensor products used in the quantum de Finetti papers are a set of unpublished lecture notes of Guichardet that treat first finite tensor products [63] and then infinite ones [62]. The approach there is to use inductive limits, which is explored in a more modern manner by Landsman [114, Sec. C.14]. For intuition's sake, we provide a direct algebraic definition below, but this is never used: theorem 3.60 states that this object is a colimit in our categories of C*-algebras. It is this property alone that is of use.

In what follows, all tensor powers are taken with respect to the same tensor product (maximal or minimal). For $m \leq n$, there exists an isometric *-homomorphism $\iota_{mn} : \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes n}$ defined on algebraic basis elements by

$$a_1 \otimes \cdots \otimes a_m \mapsto a_1 \otimes \cdots \otimes a_m \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-m \text{ times}}. \tag{3.85}$$

We define an equivalence relation on $\bigcup_{n=1}^{\infty} \mathcal{A}^{\otimes n}$ with $a \in \mathcal{A}^{\otimes m}$ equivalent to its images under ι_{mn} for all $n \geq m$. Informally, $a \sim a \otimes 1 \sim a \otimes 1 \otimes 1 \sim \ldots$.

Involution is inherited from $\mathcal{A}^{\otimes n}$, whilst multiplication is performed by placing both elements into a common space: if $a \in \mathcal{A}^{\otimes m}$ and $b \in \mathcal{A}^{\otimes n}$ for $m < n$, then

$a \cdot b := \iota_{mn}(a) \cdot b$. These are invariant under the equivalence relation and give the quotient $\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}} := \coprod_{n=1}^{\infty} \mathcal{A}^{\otimes n} / \sim$ the structure of a *-algebra, with unit given by the equivalence class of $1 \in \mathcal{A}$.

$\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}}$ also inherits a norm, since each ι_{mn} is isometric: $\|[a]\|_{\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}}} = \|a\|_{\mathcal{A}^{\otimes n}}$ for $a \in \mathcal{A}^{\otimes n}$. Under this norm, $\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}}$ fulfils all the conditions of being C*-algebra besides the fact that it is not complete. Such a space is a called a *pre-C*-algebra.* A pre-C*-algebra's norm-completion, with all operation extended continuously, is a C*-algebra [50, Lma. 2.5].

Definition 3.59 (Infinite Tensor Power)**.** *Let $\mathcal{A}$ be a C*-algebra. The* (countably) infinite tensor power of $\mathcal{A}$ *is the completion of the algebra $\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}}$ under the inherited norm above.*

It is notated by $\mathcal{A}^{\otimes\mathbb{N}}$, or $\mathcal{A}_{\text{min}}^{\otimes\mathbb{N}}$ and $\mathcal{A}_{\text{max}}^{\otimes\mathbb{N}}$ when it is necessary to distinguish whether the minimal or maximal tensor product respectively is being used.

*There are isometric *-homomorphisms $\iota_n \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes\mathbb{N}}$ given by*

$$\mathcal{A}^{\otimes n} \hookrightarrow \mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}} \hookrightarrow \mathcal{A}^{\otimes\mathbb{N}}. \tag{3.86}$$

In particular, if $a_n = \iota_{mn}(a_n)$, then $\iota_n(a_n) = \iota_m(a_m)$.

The dense subset $\mathcal{A}_{\text{loc}}^{\otimes\mathbb{N}}$ is called the set of local elements*; they are exactly those elements of $\mathcal{A}^{\otimes\mathbb{N}}$ contained in the image of ι_n for some n. Informally, these are elements of the form $a \otimes \bigotimes_{i=1}^{\infty} 1$, for some $a \in \mathcal{A}^{\otimes n}$.*

Theorem 3.60. *The space $\mathcal{A}^{\otimes\mathbb{N}}$, along with the injections $\iota_n \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes\mathbb{N}}$, is the colimit of the following* $(\mathbb{N}, \leq)$*-shaped diagram in* $\mathbf{CSt}_{\text{MIU}}$ *and in* $\mathbf{CSt}_{\text{CPU}}$

$$\mathcal{A} \xrightarrow{\iota_{12}} \mathcal{A}^{\otimes 2} \xrightarrow{\iota_{23}} \dots \xrightarrow{\iota_{(n-1)n}} \mathcal{A}^{\otimes n} \xrightarrow{\iota_{n(n+1)}} \dots \tag{3.87}$$

Explicitly, this says that given a C*-algebra $\mathcal{B}$ and *-homomorphisms (resp. completely positive, unital maps) $\phi_n \colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ for $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\phi_n = \phi_{n+1} \circ \iota_{n(n+1)}$, there exists a unique *-homomorphism (resp. completely positive, unital map) $\phi_{\mathbb{N}} \colon \mathcal{A}^{\otimes\mathbb{N}} \to \mathcal{B}$ such that for all $n \in \mathbb{N}$, $\phi_n = \phi_{\mathbb{N}} \circ \iota_n$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\iota_{12}"] \arrow[drr, bend right, "\iota_1"'] \arrow[ddrr, bend right, "\phi_1"'] & \mathcal{A}^{\otimes 2} \arrow[r, "\iota_{23}"] \arrow[dr, bend right, "\iota_2"] \arrow[ddr, bend right, "\phi_2"'] & \dots \arrow[r, "\iota_{(n-1)n}"] & \mathcal{A}^{\otimes n} \arrow[r, "\iota_{n(n+1)}"] \arrow[dl, bend left, "\iota_n"'] \arrow[ddl, bend left, "\phi_n"] & \dots \\
 & & \mathcal{A}^{\otimes \mathbb{N}} \arrow[d, dashed, "\phi_{\mathbb{N}}"] & & \\
 & & \mathcal{B} & &
\end{tikzcd} \tag{3.88}$$

The statement in $\mathbf{CSt}_{\mathrm{MIU}}$ is well-explored (and the standard motivation for defining $\mathcal{A}^{\otimes\mathbb{N}}$). We have been unable to find a source exploring the case in $\mathbf{CSt}_{\mathrm{CPU}}$. We present a proof here.

Proof of theorem 3.60. Suppose we have $\phi_n \colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ as above. Define a function

$$\begin{aligned} \phi_{\mathbb{N}} \colon \mathcal{A}_{\mathrm{loc}}^{\otimes\mathbb{N}} &\to \mathcal{B} \\ \iota_n(a_n) &\mapsto \phi_n(a_n) \end{aligned} \tag{3.89}$$

for any $n \in \mathbb{N}$ and $a_n \in \mathcal{A}^{\otimes n}$.

Each ι_n is injective so if $\iota_n(a_n) = \iota_m(a_m)$ for $m \leq n$, then $a_n = \iota_{mn}(a_m)$ and

$$\phi_n(a_n) = \phi_n(\iota_{mn}(a_m)) = \phi_m(a_m). \tag{3.90}$$

Thus, $\phi_{\mathbb{N}}$ is well-defined. It is the unique set-function defined on local elements with $\iota_n \circ \phi_{\mathbb{N}} = \phi_n$ for all $n \in \mathbb{N}$.

The map $\phi_{\mathbb{N}}$ is bounded on $\mathcal{A}_{\mathrm{loc}}^{\otimes\mathbb{N}}$ since for any $a = \iota_n(a_n) \in \mathcal{A}_{\mathrm{loc}}^{\otimes\mathbb{N}}$:

$$\|\phi_{\mathbb{N}}(a)\|_{\mathcal{B}} = \|\phi_n(a_n)\|_{\mathcal{B}} \leq \|\phi_n\| \, \|a_n\|_{\mathcal{A}^{\otimes n}} = \|a_n\|_{\mathcal{A}^{\otimes n}} = \|a\|_{\mathcal{A}^{\otimes\mathbb{N}}} \tag{3.91}$$

since positive maps (which both *-homomorphisms and completely positive maps are) have unit norm (propositions 3.6 and 3.20).

As such, by lemma 3.42, this map extends to a bounded linear map on $\mathcal{A}^{\otimes\mathbb{N}}$. It is clear that the *-homomorphism structure extends straightforwardly from continuity of multiplication and involution.

It remains only to show that $\phi_{\mathbb{N}}$ is completely positive.

Take $B \in M^n\left(\mathcal{A}_{\mathrm{loc}}^{\otimes\mathbb{N}}\right)$. Since B has finitely many entries, and each entry is of the form $\iota_{m_{i,j}}(b_{i,j})$, we may write $B = (\iota_M(\beta_{i,j}))$ for some $M \in \mathbb{N}$ and $\beta_{i,j} \in \mathcal{A}^{\otimes M}$. Thus, $B = \iota_M^{(n)}\left(\hat{B}\right)$, for $\hat{B} := (\beta_{i,j}) \in M^n\left(\mathcal{A}^{\otimes M}\right)$. Then, $B^*B = \iota_M^{(n)}\left(\hat{B}^*\hat{B}\right)$.

Thus, $\phi^{(n)}_{\mathbb{N}}\left(B^*B\right) = \phi^{(n)}_{M}\left(\hat{B}^*\hat{B}\right) \in \mathcal{B}_+$.

Now suppose $A \in \left(M^n\left(\mathcal{A}^{\otimes\mathbb{N}}\right)\right)_+$ and as such there exists $B \in M^n\left(\mathcal{A}^{\otimes\mathbb{N}}\right)$ with $A = B^*B$. Let $(B_n) \subset M^n\left(\mathcal{A}^{\otimes\mathbb{N}}_{\text{loc}}\right)$ be a sequence with $B_n \to B$ and $B_n^*B_n \to A$ (for existence of such a sequence, see lemma 3.61 below).

$\mathcal{B}_+$ is closed, and $\phi_{\mathbb{N}}(B_n^*B_n) \in \mathcal{B}_+$ for all n by the above. Since $\phi_{\mathbb{N}}$ is bounded, $\lim_{n\to\infty}\phi_{\mathbb{N}}\left(B_n^*B_n\right) = \phi_{\mathbb{N}}\left(A\right)$ is in $\mathcal{B}_+$ as desired. □

The existence of the sequence B_n in the second to last step is justified by the following lemma:

Lemma 3.61. *Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{A}_p \subset \mathcal{A}$ be a dense subset of $\mathcal{A}$. Then $M^n(\mathcal{A}_p)$ is dense in $M^n(\mathcal{A})$.*

Proof. Given a matrix $A = (a_{ij})_{1\leq i,j\leq n}$ in $M^n(\mathcal{A})$, we have that $A = \sum_{i,j} a_{ij}\otimes E_{ij}$, where $E_{ij} \in M^n(\mathbb{C})$ is the matrix with a 1 in the $(i,j)^{\text{th}}$ place and zeros elsewhere. Note that $\|E_{ij}\|_{M^n(\mathbb{C})} = 1$. A cross-norm on $M^n(\mathcal{A})$ will then give

$$\|A\| = \left\|\sum_{i,j} a_{ij}\otimes E_{ij}\right\| \leq \sum_{i,j}\|a_{ij}\|_{\mathcal{A}}\|E_{ij}\|_{M^n(\mathbb{C})} = \sum_{i,j}\|a_{ij}\|_{\mathcal{A}}. \tag{3.92}$$

Thus, convergence element-wise to a_{ij} with elements of $\mathcal{A}_p$ is sufficient for convergence of matrices. □

An important corollary of theorem 3.71 is building product states on the infinite tensor product from compatible states on local elements. The classical Kolmogorov extension will be treated in chapter 4, and shows that measures on $X^{\mathbb{N}}$ are constructed from stitching together compatible measures on the finite products of X. Our quantum Kolmogorov extension theorem says the same for states on the infinite tensor product.

Corollary 3.62 (Quantum Kolmogorov Extension Theorem)**.** *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of states $\rho_n \in S(\mathcal{A}^{\otimes n})$ such that for all $m \leq n$, $\rho_m = \rho_n \circ \iota_{mn}$. There is a unique state $\rho \in S\left(\mathcal{A}^{\otimes\mathbb{N}}\right)$ such that $\rho_n = \rho\circ\iota_n$ for all $n \in \mathbb{N}$.*

Proof. This follows from theorem 3.60 in $\mathbf{CSt}_{\text{CPU}}$ with $\phi_n = \rho_n$ and $\mathcal{B} = \mathbb{C}$. □

By instantiating corollary 3.62 using power states, we can form the infinite power states.

Corollary 3.63 (Infinite Power State)**.** *Let* $\rho \in S(\mathcal{A})$ *be a state on a C*-algebra. There exists a state* $\rho^{\otimes \mathbb{N}} \in S\left(\mathcal{A}^{\otimes \mathbb{N}}\right)$ *such that for any* $\iota_n(a) \in \mathcal{A}^{\otimes \mathbb{N}}_{\mathrm{loc}}$

$$\rho^{\otimes \mathbb{N}} \left(\iota_n(a)\right) = \rho^{\otimes n} \left(a\right). \tag{3.93}$$

In particular, if $a = \bigotimes_{n \in \mathbb{N}} a_n$*, with* $a_n \neq 1$ *for only finitely many* n*, then*

$$\rho^{\otimes \mathbb{N}} \left(a\right) = \prod_{n \in \mathbb{N}} \rho\left(a_n\right). \tag{3.94}$$

Proof. We may construct $\rho^{\otimes \mathbb{N}}$ via corollary 3.62 with the power states $\rho^{\otimes n} \in S(\mathcal{A}^{\otimes n})$. $\square$

Corollary 3.62 tells us that states on the infinite tensor power C*-algebra $\mathcal{A}^{\otimes \mathbb{N}}$ are exactly compatible sequences of states on finite C*-algebra tensor powers. As such, we may specify properties of a state $\rho \in S\left(\mathcal{A}^{\otimes \mathbb{N}}\right)$ by its restrictions to local elements $\rho_n := \rho \circ \iota_n$. In the same way as classically, then, this is how we define exchangeability.

Definition 3.64 (Exchangeable Quantum States)**.** *Let* $\mathcal{A}$ *be a C*-algebra. A state* $\rho \in S\left(\mathcal{A}^{\otimes \mathbb{N}}\right)$ *on an infinite tensor power of* $\mathcal{A}$ *is called* exchangeable *if its restrictions to local elements* $\rho_n \in S(\mathcal{A}^{\otimes n})$ *are symmetric.*

Recall from definition 3.57 that a state $\rho_n \in S(\mathcal{A}^{\otimes n})$ is symmetric if for every permutation $\sigma \in \mathcal{S}_n$, $\rho_n \circ \eta_\sigma = \rho_n$.

The question of classification of these states is exactly what quantum de Finetti theorems set out to prove.

3.4.3 Two Quantum de Finetti Theorems

In this thesis, we are concerned with two quantum de Finetti theorems. Theorem 3.65, from Størmer, classifies symmetric states on $\mathcal{A}^{\otimes \mathbb{N}}_{\mathrm{min}}$, whilst theorem 3.66, from Hulanicki and Phelps, does the same for $\mathcal{A}^{\otimes \mathbb{N}}_{\mathrm{max}}$. It is these two that will be situated categorically in section 3.5.

It is worth noting that a number of de Finetti theorems have been shown in quantum contexts. Closely related are those of Hudson and Moody [78, 79], whilst there are also extensions to more complicated quantum groups [e.g. 19] and quantum stochastic processes [e.g. 30]. There also exist theorems in free probability [110], and in generalised probability [11]. Future work would be to explore extensions in these contexts.

Recent work by Fritz and Lorenzin has established a de Finetti theorem for pre-C*-algebras [50]. From this, they deduce both the minimal and maximal tensor product de Finetti theorems for C*-algebras using the tools of their involutive Markov categories, a string-diagrammatic axiomatisation of quantum probability derived from Markov categories. This approach is insightful, and deals with the properties of de Finetti objects, that is, limits of diagrams like those discussed in this thesis, in this abstract setting. Since the algebraic tensor product of C*-algebras with a norm attached is a pre-C*-algebra, there is some hint here of future work on a categorical de Finetti theorem for a general (cross-norm derived) monoidal structure on $\mathbf{CSt}_{\mathrm{CPU}}$.

The theorem of Størmer's discussed in the introduction to this chapter lays out the form that these quantum de Finetti theorems take:

Theorem 3.65 (Minimal Tensor Product Quantum de Finetti Theorem [147, Thm. 2.8])**.** *Let* $I_{\min}(\mathcal{A}) := \left\{\rho \in S\left(\mathcal{A}^{\otimes\mathbb{N}}_{\min}\right) : \rho \textit{ is exchangeable}\right\}$ *for a C*-algebra* $\mathcal{A}$*. There exists a homeomorphism* $\partial I_{\min}(\mathcal{A}) \cong S(\mathcal{A})$ *which extends by Choquet's theorem to an affine homeomorphism*

$$I_{\min}(\mathcal{A}) \cong \mathcal{R}(S(\mathcal{A})). \tag{3.95}$$

Recall that $\partial\mathbf{W}$, for $\mathbf{W} \in \mathbf{ConvCH}$, is the set of extreme points of $\mathbf{W}$, definition 2.19. For Choquet's theorem see [128, Chap. 4]. It says that under certain conditions, convex spaces are isomorphic to the spaces of measures supported on their extreme points, via integration over those extreme points.

The bijection $\partial I_{\min}(\mathcal{A}) \cong S(\mathcal{A})$ is given by $\rho \in S(\mathcal{A}) \mapsto \rho^{\otimes\mathbb{N}} \in \partial I_{\min}(\mathcal{A})$ and as such the isomorphism of eq. (3.95), states that for all $\rho \in I_{\min}(\mathcal{A})$, there exists a measure $\mu \in \mathcal{R}(S(\mathcal{A}))$ such that for all $a \in \mathcal{A}_{\min}^{\otimes\mathbb{N}}$

$$\omega(a) = \int_{\rho\in S(\mathcal{A})} \rho^{\otimes\mathbb{N}}(a)\,\mathrm{d}\mu. \tag{3.96}$$

Naturally, one might ask, does a similar theorem hold for the maximal tensor product? Hulanicki and Phelps in [80], concurrently to Størmer, and Landsman in [114], answer in the affirmative. Not just similar, but identical.

Theorem 3.66 (Maximal Tensor Product Quantum de Finetti Theorem [80, Theorem 4.1, 114, Theorem 8.6])**.** *Let $\mathcal{A}$ be a C*-algebra. Let*

$$I_{\max}(\mathcal{A}) := \left\{\rho \in S\left(\mathcal{A}_{\max}^{\otimes\mathbb{N}}\right) : \rho \text{ is exchangeable}\right\}. \tag{3.97}$$

There exists a homeomorphism $\partial I_{\max}(\mathcal{A}) \cong S(\mathcal{A})$. Via the same reasoning as theorem 3.65, this extends to an affine homeomorphism

$$I_{\max}(\mathcal{A}) \cong \mathcal{R}(S(\mathcal{A})). \tag{3.98}$$

In particular, we have the following

Theorem 3.67. *For a C*-algebra $\mathcal{A}$, $I_{\min}(\mathcal{A}) \cong I_{\max}(\mathcal{A})$.*

This result follows immediately from the above, but could not be found in print. It is surprising that even non-nuclear C*-algebras have identical exchangeable states on both their minimal and maximal infinite tensor products.

3.5 Main Results: Quantum de Finetti Theorems as Categorical Limits

3.5.1 Sketch of the Result

We now have the framework required to state and prove the main results of this chapter.

The idea, as stated in section 3.1.2, is to construct an exchangeable state as a cocone over a diagram in a category of C*-algebras, and to show that the de Finetti classification defines a colimit of this diagram. Since the study of quantum systems situates itself in $\mathbf{CSt}_{\mathrm{CPU}}$, this will be where we have a colimit, and we will use the various properties of **ConvCH** and $\mathbf{CSt}_{\mathrm{PU}}$ to construct it there.

Recall from section 3.1.2 that an exchangeable sequence of states on a Hilbert space $\mathcal{H}$ is given by a $\mathbb{C}$-apex cone over diag. (3.3):

$$\begin{tikzcd}
\mathcal{H} \arrow[loop above] & \mathcal{H}^{\otimes 2} \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above] & \mathcal{H}^{\otimes 3} \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \mathcal{H}^{\otimes n} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] \\
 & & & \mathbb{C} \arrow[ulll, bend left] \arrow[ull, bend left] \arrow[ul, bend left] \arrow[ur, bend right] & &
\end{tikzcd} \tag{3.99}$$

We now know that the appropriate place to draw this diagram is in $\mathbf{CSt}_{\mathrm{CPU}}$. Quantum channels $\mathbb{C} \to \mathcal{H}^{\otimes n}$ correspond to completely positive, unital maps $\mathcal{B}(\mathcal{H}^{\otimes n}) \to \mathbb{C}$. All the maps in the diagram are generated by partial traces and permutations of factors. The partial trace $\mathrm{tr}_{n(n-1)} \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$ corresponds to the map

$$\begin{aligned} \mathcal{B}\big(\mathcal{H}^{\otimes n-1}\big) &\to \mathcal{B}\big(\mathcal{H}^{\otimes n}\big) \\ \Phi &\mapsto \Phi \otimes \mathrm{id}_{\mathcal{H}} \end{aligned} \tag{3.100}$$

whilst a permutation $\sigma \in \mathcal{S}_n$ would act on $\mathcal{B}(\mathcal{H}^{\otimes n})$ by precomposing a given operator with the braiding map of σ^{-1} on the tensor power of $\mathcal{H}$. This was how η_σ was defined in definition 3.57.

As such, the appropriate diagram to represent a single exchangeable state on a quantum system on the Hilbert space $\mathcal{H}$ in $\mathbf{CSt}_{\mathrm{CPU}}$ would be a cocone as below:

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) \arrow[loop above] \arrow[r, shift left, "-\otimes \mathrm{id}_{\mathcal{H}}"] \arrow[r, shift right, "\mathrm{id}_{\mathcal{H}}\otimes -"'] \arrow[drr, bend right] & \mathcal{B}(\mathcal{H}^{\otimes 2}) \arrow[loop above] \arrow[loop above, "\eta_{\mathrm{swap}}"] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[dr, bend right] & \cdots \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & \mathcal{B}(\mathcal{H}^{\otimes n}) \arrow[loop above, "\eta_\sigma"] \arrow[loop above] \arrow[loop above] \arrow[loop above, "\eta_\tau"] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] \arrow[dl, bend left] & \cdots \\
 & & \mathbb{C} & &
\end{tikzcd} \tag{3.101}$$

This diagram then is naturally generalised in two ways: by replacing $\mathcal{B}(\mathcal{H})$ with a general C*-algebra $\mathcal{A}$, and replacing the states $\mathcal{B}(\mathcal{H}^{\otimes n}) \to \mathbb{C}$ with general compatible morphisms $\mathcal{A}^{\otimes n} \to \mathcal{B}$.

An exchangeable state on a general C*-algebra $\mathcal{A}$ is thus a cocone with apex $\mathbb{C}$ over the diagram $\mathbf{Inj} \to \mathbf{CSt}_{\mathrm{CPU}}$ which takes each natural number n to some well-chosen tensor power $\mathcal{A}^{\otimes n}$ and the injection $\tau\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ to the map $\eta_\tau\colon \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes n}$ which takes $a_1 \otimes \cdots \otimes a_m \mapsto \alpha_1 \otimes \cdots \otimes \alpha_n$ for

$$\alpha_k = \begin{cases} a_{\tau^{-1}(k)} & \text{if } k \in \operatorname{im}(\tau) \\ 1_{\mathcal{A}} & \text{otherwise.} \end{cases} \tag{3.102}$$

We may depict such a cocone with the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[loop above] \arrow[r, shift left, "-\otimes 1_{\mathcal{A}}"] \arrow[r, shift right] \arrow[drr, bend right] & \mathcal{A}^{\otimes 2} \arrow[loop above] \arrow[loop above, "\eta_{\mathrm{swap}}"] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[dr, bend right] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{A}^{\otimes n} \arrow[loop above, "\eta_\sigma"] \arrow[loop above] \arrow[loop above] \arrow[loop above, "\eta_\tau"] \arrow[dl, bend left] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \cdots \\
& & \mathbb{C} & &
\end{tikzcd} \tag{3.103}$$

A *generalised* or *parametrised exchangeable state* on a C*-algebra $\mathcal{A}$ is a general cocone of this diagram with some apex $\mathcal{B}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[loop above] \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, bend right] & \mathcal{A}^{\otimes 2} \arrow[loop above] \arrow[loop above] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[dr, bend right] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{A}^{\otimes n} \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[dl, bend left] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \cdots \\
& & \mathcal{B} & &
\end{tikzcd} \tag{3.104}$$

As described in the special case for $\mathcal{H}$ in section 3.1, our categorical quantum de Finetti theorem says that this diagram has a colimit, which is to say a universal cocone: a C*-algebra $\mathrm{QdF}(\mathcal{A})$ with maps $\epsilon_n\colon \mathcal{A}^{\otimes n} \to \mathrm{QdF}(\mathcal{A})$ commuting with the exchangeability diagram (i.e. for any $\tau \in \mathbf{Inj}(m, n)$, $\epsilon_m = \epsilon_n \circ \eta_\tau$) such that if we have a parametrised exchangeable state like in diag. (3.104) there exists a unique mediating morphism $\mathrm{QdF}(\mathcal{A}) \to \mathcal{B}$, indeed a parametrised state on $\mathrm{QdF}(\mathcal{A})$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[loop above] \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, bend right] \arrow[ddrr, bend right] & \mathcal{A}^{\otimes 2} \arrow[loop, out=150, in=100, distance=2em] \arrow[loop, out=80, in=30, distance=2em] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[dr, bend right] \arrow[ddr, bend right] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{A}^{\otimes n} \arrow[loop, out=160, in=130, distance=2em] \arrow[loop, out=130, in=100, distance=2em] \arrow[loop, out=100, in=80, distance=2em] \arrow[loop, out=80, in=50, distance=2em] \arrow[loop, out=50, in=20, distance=2em] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[dl, bend left] \arrow[ddl, bend left] & \cdots \\
 & & \mathrm{QdF}(\mathcal{A}) \arrow[d, dashed] & & \\
 & & \mathcal{B}. & &
\end{tikzcd} \tag{3.105}$$

The choice of $\mathrm{QdF}(\mathcal{A})$ is informed by the traditional quantum de Finetti theorems: we may just as well consider, as indeed we will to prove the result, this diagram composed with the state-space functor. The composed diagram

$$\begin{tikzcd}
S(\mathcal{A}) \arrow[loop above] & S(\mathcal{A}^{\otimes 2}) \arrow[loop, out=150, in=100, distance=2em] \arrow[loop, out=80, in=30, distance=2em] \arrow[l, shift left] \arrow[l, shift right] & S(\mathcal{A}^{\otimes 3}) \arrow[loop, out=160, in=130, distance=2em] \arrow[loop, out=130, in=100, distance=2em] \arrow[loop, out=100, in=80, distance=2em] \arrow[loop, out=80, in=50, distance=2em] \arrow[loop, out=50, in=20, distance=2em] \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & S(\mathcal{A}^{\otimes n}) \arrow[loop, out=160, in=130, distance=2em] \arrow[loop, out=130, in=100, distance=2em] \arrow[loop, out=100, in=80, distance=2em] \arrow[loop, out=80, in=50, distance=2em] \arrow[loop, out=50, in=20, distance=2em] \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2]
\end{tikzcd} \tag{3.106}$$

has a cone with apex $I(\mathcal{A})$, the space of exchangeable states on $\mathcal{A}$, taking a state $\rho \in I(\mathcal{A}) \subset S\big(\mathcal{A}^{\otimes \mathbb{N}}\big)$ to its local restrictions $\rho_n \in S(\mathcal{A}^{\otimes n})$:

$$\begin{tikzcd}
S(\mathcal{A}) \arrow[loop above] & S(\mathcal{A}^{\otimes 2}) \arrow[loop, out=150, in=100, distance=2em] \arrow[loop, out=80, in=30, distance=2em] \arrow[l, shift left] \arrow[l, shift right] & S(\mathcal{A}^{\otimes 3}) \arrow[loop, out=160, in=130, distance=2em] \arrow[loop, out=130, in=100, distance=2em] \arrow[loop, out=100, in=80, distance=2em] \arrow[loop, out=80, in=50, distance=2em] \arrow[loop, out=50, in=20, distance=2em] \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & S(\mathcal{A}^{\otimes n}) \arrow[loop, out=160, in=130, distance=2em] \arrow[loop, out=130, in=100, distance=2em] \arrow[loop, out=100, in=80, distance=2em] \arrow[loop, out=80, in=50, distance=2em] \arrow[loop, out=50, in=20, distance=2em] \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \\
 & & & I(\mathcal{A}). \arrow[ulll, bend left] \arrow[ull, bend left] \arrow[ul, bend left] \arrow[ur, bend right] & &
\end{tikzcd} \tag{3.107}$$

This space is a very reasonable candidate for the limit of such a diagram and indeed we will show it is the limit. The suggestion of the isomorphisms of theorems 3.65 and 3.66, which say that for both the maximal and minimal tensor products $I(\mathcal{A}) \cong \mathcal{R}(S(\mathcal{A})) \cong S(C(S(\mathcal{A})))$, is then that the corresponding colimit $\mathrm{QdF}(\mathcal{A})$, reflected by S, should be $CS(\mathcal{A})$. In this section, it is exactly this, that $CS(\mathcal{A})$ is the colimit of the diagram, that is what is proved.

3.5.2 The State-Space Functor and (Co)Limits

This section is as in our QPL paper [144]. The state-space functor $S\colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{ConvCH}$, alongside the favourable properties shown in [58], also behaves nicely with limits.

Theorem 3.68. *The state-space functor* $S\colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{ConvCH}$ *preserves and reflects limits.*

Note that this means that a diagram of C*-algebras, $\mathcal{A}_{-}\colon \mathcal{I} \to \mathbf{CSt}_{\mathrm{PU}}$ has colimit given by $\{\mathcal{A}_i \to \mathcal{B}\}_{i\in\mathcal{I}}$ if and only if the corresponding cone $\{S(\mathcal{B}) \to S(\mathcal{A}_i)\}_{i\in\mathcal{I}}$ is the limit of the diagram $\mathcal{I}^{\mathrm{op}} \overset{\mathcal{A}_{-}}{\to} \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \overset{S}{\to} \mathbf{ConvCH}$.

Proof. S is full and faithful by theorem 3.25, and as such it must reflect limits and colimits.

Now let us recall the factorisation of the hom-functor $\mathbf{CSt}_{\mathrm{PU}}(-,\mathbb{C})\colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{Set}$ in diag. (3.26), through S and two monadic forgetful functors:

$$\begin{tikzcd}
\mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \arrow[r, two heads, "S"'] \arrow[rrr, bend left, "\mathbf{CSt}_{\mathrm{PU}}(-{,}\mathbb{C})"] & \mathbf{ConvCH} \arrow[r, "U_{\mathbf{ConvCH}}"'] & \mathbf{CH} \arrow[r, "U_{\mathbf{CH}}"'] & \mathbf{Set}.
\end{tikzcd} \tag{3.108}$$

Monadic functors create limits, and hom-functors preserve them. As such, if a colimit exists in $\mathbf{CSt}_{\mathrm{PU}}$, it is preserved via $\mathbf{CSt}_{\mathrm{PU}}(-,\mathbb{C})$ into a limit in $\mathbf{Set}$. This limit then creates a limit in $\mathbf{CH}$ via $U_{\mathbf{CH}}$, which then creates another limit in $\mathbf{ConvCH}$ via $U_{\mathbf{ConvCH}}$. The commutativity of the above diagram then ensures that this limit is exactly the image of the colimit in $\mathbf{CSt}_{\mathrm{PU}}$ under S, as desired. □

This factorisation also allows us to construct our limits in some precise sense "pointwise", which is to say that a limit in $\mathbf{ConvCH}$ is exactly a $\mathbf{Set}$-limit, which then can lift to $\mathbf{CSt}_{\mathrm{PU}}$.

Lemma 3.69 (Pointwise limits are limits in **ConvCH**)**.** *Let* $\mathcal{I}$ *be a category. Consider a collection* $\{\mathbf{W}_i\}_{i\in\mathcal{I}}$ *of spaces in* **ConvCH** *with affine continuous maps* $\{\zeta_k\colon \mathbf{W}_i \to \mathbf{W}_j\}_{k\in\mathcal{I}(i,j)}$*, that is, a diagram* $\mathbf{W}_{-}\colon \mathcal{I} \to \mathbf{ConvCH}$.

Let $\mathbf{W}$ *be a cancellative, convex, compact Hausdorff space with morphisms* $\varrho_i \colon \mathbf{W} \to \mathbf{W}_i$ *for each* $i \in \mathcal{I}$*, satisfying the following properties:*

1. $\{\varrho_i \colon \mathbf{W} \to \mathbf{W}_i\}_{i \in \mathcal{I}}$ *is a cone over the diagram. In other words, for all* $\zeta_k \colon \mathbf{W}_i \to \mathbf{W}_j$ *in the diagram,* $\varrho_i = \varrho_j \circ \zeta_k$*.*
2. *Pointwise limit property illustrated in diag. (3.109): For any collection of elements* $(w_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \mathbf{W}_i$ *which are compatible with the diagram, in the sense that for all* $\zeta_k \colon \mathbf{W}_i \to \mathbf{W}_j$*,* $w_i = \zeta_k(w_j)$*, there is a unique element* $w \in \mathbf{W}$ *with* $w_i = \varrho_i(w)$*.*

$$\begin{tikzcd}
 & & \mathbf{W}_i \arrow[dd, "\zeta_k"] \\
\{*\} \arrow[r, dashed, "\exists ! w"] \arrow[urr, bend left, "w_i"] \arrow[drr, bend right, "w_j"'] & \mathbf{W} \arrow[ur, "\varrho_i"] \arrow[dr, "\varrho_j"] & \\
 & & \mathbf{W}_j
\end{tikzcd} \tag{3.109}$$

Then $\mathbf{W}$ *is the limit of the diagram in* **CH** *and* **ConvCH***.*

Proof. Suppose we have a diagram $\mathbf{W}_-$ as above, and a cone $\{\varrho_i \colon \mathbf{W} \to \mathbf{W}_i\}$ for some $\mathbf{W} \in \mathbf{ConvCH}$ which has the pointwise limit property.

Then for any cone in **Set** over the diagram $W_- := U_{\mathbf{CH}} \circ U_{\mathbf{ConvCH}} \circ \mathbf{W}_- \colon \mathcal{I} \to \mathbf{Set}$, which is a compatible collection of functions $\{f_i \colon V \to W_i\}_{i \in \mathcal{I}}$, for $W_i = U_{\mathbf{CH}} \circ U(\mathbf{W}_i)$ the underlying set of $\mathbf{W}_i$, we may construct the unique mediating map $f \colon V \to W$ by taking $f(a)$ equal to the unique $w \in W$ corresponding to $\{f_i(a) \in \mathbf{W}_i\}_{i \in \mathcal{I}}$ under the pointwise limit property.

$$\begin{tikzcd}
 & & W_i \arrow[dd, "\zeta_k"] \\
V \arrow[r, dashed, "\exists ! f"] \arrow[urr, bend left, "\varphi_i"] \arrow[drr, bend right, "\varphi_j"'] & W \arrow[ur, "\varrho_i"] \arrow[dr, "\varrho_j"] & \\
 & & W_j
\end{tikzcd} \tag{3.110}$$

Then the construction above witnesses W as the limit of the diagram in **Set** and the monadicity of U and U' creates $\mathbf{W}$ as the limit in **CH** and **ConvCH** as desired. $\square$

Informally, this lemma shows that we get categorical limits for free when we operate pointwise in **ConvCH**. In particular the statements of theorems 3.65 and 3.66 are all describing the pointwise limit property for an appropriate diagram, and so when moved into **ConvCH** with the functor S give rise to statements as limits, without having to wrangle with the technicalities of showing continuity or affineness.

In the case that the spaces in the statement of lemma 3.69 are in fact state spaces $\mathbf{W}_i = S(\mathcal{A}_i)$ and $\mathbf{W} = S(\mathcal{B})$, as they will be in our categorical quantum de Finetti theorems, the lemma can be situated in $\mathbf{CSt}_{\mathrm{PU}}$. Since S is full and faithful, for each $\zeta_k \colon S(\mathcal{A}_i) \to S(\mathcal{A}_j)$ there is a corresponding $\psi_k \colon \mathcal{A}_j \to \mathcal{A}_i$, and each $\varrho_i \colon S(\mathcal{B}) \to S(\mathcal{A}_i)$ corresponds to a $\phi_i \colon \mathcal{A}_i \to \mathcal{B}$. The first property of the lemma says that these ϕ_is are a cocone of the diagram of ψ_ks. The pointwise limit property then says for each collection of states $\{\rho_i \colon \mathcal{A}_i \to \mathbb{C}\}_{i \in \mathcal{I}}$ which commute with the ψ_ks, there exists a unique state $\rho \in S(\mathcal{B})$ such that $\rho_i = \rho \circ \phi_i$ for all $i \in \mathcal{I}$:

$$\begin{tikzcd}
\mathcal{A}_i \arrow[dr, "\phi_i"] \arrow[drr, bend left, "\rho_i"] & & \\
& \mathcal{B} \arrow[r, dashed, "\exists!\rho"] & \mathbb{C} \\
\mathcal{A}_j \arrow[uu, "\psi_k"] \arrow[ur, "\phi_j"'] \arrow[urr, bend right, "\rho_j"'] & &
\end{tikzcd} \tag{3.111}$$

Corollary 3.70. *$\mathcal{B}$ is a genuine colimit of the diagram of $\mathcal{A}_i$s and ψ_ks in* $\mathbf{CSt}_{\mathrm{PU}}$.

Proof. Lemma 3.69 tells us that $S(\mathcal{B})$ is the limit of the appropriate diagram in **ConvCH**. $S \colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{ConvCH}$ reflects limits by theorem 3.68, so $\mathcal{B}$ is the colimit in $\mathbf{CSt}_{\mathrm{PU}}$. □

As such, we may replace $\mathbb{C}$ in diag. (3.111) with any C*-algebra $\mathcal{A}'$, and the states ρ_i with general positive, unital maps ϑ_i.

$$\begin{tikzcd}
\mathcal{A}_i \arrow[dr, "\phi_i"] \arrow[drr, bend left, "\vartheta_i"] & & \\
& \mathcal{B} \arrow[r, dashed, "\exists!\vartheta"] & \mathcal{A}'. \\
\mathcal{A}_j \arrow[uu, "\psi_k"] \arrow[ur, "\phi_j"'] \arrow[urr, bend right, "\vartheta_j"'] & &
\end{tikzcd} \tag{3.112}$$

3.5.3 Categorical Quantum Kolmogorov Extension Theorem

With the story as laid out so far in this chapter, once the right choices are made for diagrams, the de Finetti limits jump out straightforwardly. All the heavy lifting of proof has already been done; now, we reap the rewards. First we establish a categorical quantum Kolmogorov extension theorem.

Theorem 3.71 (Categorical Quantum Kolmogorov Extension Theorem)**.** *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $\mathcal{A}^{\otimes n}$ *be the tensor powers of* $\mathcal{A}$ *under a fixed choice of either the maximal or minimal tensor product. Let* $\mathcal{A}^{\otimes \mathbb{N}}$ *be the corresponding infinite tensor power. The limit of the* $(\mathbb{N}, \leq)^{\mathrm{op}}$*-shaped diagram of state spaces in* **ConvCH**

$$S(\mathcal{A}) \xleftarrow{-\circ\iota_{12}} S(\mathcal{A}^{\otimes 2}) \xleftarrow{-\circ\iota_{23}} \dots \xleftarrow{-\circ\iota_{(n-1)n}} S(\mathcal{A}^{\otimes n}) \xleftarrow{-\circ\iota_{n(n+1)}} \dots \tag{3.113}$$

is $S\left(\mathcal{A}^{\otimes \mathbb{N}}\right)$ *with the inclusions* $-\circ\iota_n \colon S\left(\mathcal{A}^{\otimes \mathbb{N}}\right) \to S(\mathcal{A}^{\otimes n})$*.*

Recall from definition 3.59 that for $m \leq n$, $\iota_{mn} \colon \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes n}$ takes $a \in \mathcal{A}^{\otimes m}$ to $a \otimes \underbrace{1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}}_{n-m \text{ times}}$ on algebraic basis elements, and informally $\iota_n(a) = a \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} \otimes \cdots \in \mathcal{A}^{\otimes \mathbb{N}}$.

Proof. $\mathcal{A}^{\otimes \mathbb{N}}$ is the colimit in $\mathbf{CSt}_{\mathrm{PU}}$ of the diagram $\mathcal{A} \xrightarrow{\iota_{12}} \mathcal{A}^{\otimes 2} \xrightarrow{\iota_{23}} \mathcal{A}^{\otimes 3} \to \dots$. The image of this diagram under S is diag. (3.113), and this functor preserves limits (or turns colimits into limits) by theorem 3.68. The result follows. ☐

The categorical limit versions of traditional theorems can be understood as introducing parameterisation. Where corollary 3.62 says that a compatible sequence of quantum states $\rho_i \colon \mathcal{A}_i \to \mathbb{C}$ gives rise to a state $\rho \colon \mathcal{A}^{\otimes \mathbb{N}} \to \mathbb{C}$ on the infinite system, theorem 3.71 says that this remains true if we do not just have one state, but instead parameterise this sequence of states according to affine and continuous maps $\theta \mapsto \rho_i(- \mid \theta)$ out of some cancellative, convex, compact Hausdorff topological space $\mathbf{W} \in$ **ConvCH**, giving rise to an affine and continuous parametrisation of states on the infinite system $\theta \mapsto \rho(- \mid \theta) \in S\left(\mathcal{A}^{\otimes \mathbb{N}}\right)$.

$$\begin{tikzcd}
S(\mathcal{A}) & S(\mathcal{A}^{\otimes 2}) \arrow[l, "-\circ\iota_{12}"'] & \dots \arrow[l, "-\circ\iota_{23}"'] & S(\mathcal{A}^{\otimes n}) \arrow[l, "-\circ\iota_{(n-1)n}"'] & \dots \arrow[l, "-\circ\iota_{n(n+1)}"'] \\
 & & & S\left(\mathcal{A}^{\otimes\mathbb{N}}\right) \arrow[u, "-\circ\iota_n"] \arrow[ulll, bend left] \arrow[ull, bend left] & \\
 & & & & \mathbf{W}. \arrow[ul, dashed, "\Theta"'] \arrow[uul, bend right, "\Theta_n"'] \arrow[uulll, bend left, "\Theta_2"] \arrow[uullll, bend left, "\Theta_1"]
\end{tikzcd} \tag{3.114}$$

3.5.4 Quantum de Finetti Theorems as Categorical Limits

Here we prove the main result of this chapter, theorem 3.72, the categorical quantum de Finetti theorem in the category $\mathbf{CSt}_{\text{CPU}}$ of completely positive, unital maps. To do this, however, we go through $\mathbf{CSt}_{\text{PU}}$, the category of C*-algebras and **all positive, unital maps**.

Recall definition 2.1: the category **Inj**, which has as its objects the natural numbers $n \in \mathbb{N}$. The morphisms in $\mathbf{Inj}(m,n)$, for $m, n \in \mathbb{N}$, are the injections $\{1,\dots,m\} \hookrightarrow \{1,\dots,n\}$, with composition in **Inj** given by function composition.

If $n \geq m$, then $\mathbf{Inj}(n,m) = \varnothing$. This category is generated by $\mathcal{S}_n = \mathbf{Inj}(n,n)$, the collection of all symmetric groups of permutations on $\{1,\dots,n\}$, and by the identity injections $\{1,\dots,n\} \to \{1,\dots,n+1\}$.

Theorem 3.72 (Categorical Quantum de Finetti Theorem)**.** *Let $\mathcal{A}$ be a C*-algebra and* **Inj** *be as above. Let us define two diagrams $\mathcal{D}_{\text{min}}, \mathcal{D}_{\text{max}} \colon \mathbf{Inj} \to \mathbf{CSt}_{\text{CPU}}$, with $\mathcal{D}_{\text{min}}(n) = \mathcal{A}^{\otimes n}_{\text{min}}$ and $\mathcal{D}_{\text{max}}(n) = \mathcal{A}^{\otimes n}_{\text{max}}$.*

On morphisms, each diagram takes a permutation $\sigma \in \mathcal{S}_n$ to the braiding map $\eta_\sigma = \mathcal{A}^{\otimes\left(\sigma^{-1}\right)} \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$. The identity injection $\{1,\dots,n\} \hookrightarrow \{1,\dots,n+1\}$ is taken to the isometry $\iota_{n(n+1)} = - \otimes 1_{\mathcal{A}}$.

The colimit of the diagram $\mathcal{D}$ in $\mathbf{CSt}_{\text{CPU}}$ is $\{\epsilon_n \colon \mathcal{A}^{\otimes n} \to CS(\mathcal{A})\}_{n\in\mathbb{N}}$ with the appropriate tensor product in the domain, for

$$\epsilon_n(a) = \lambda\rho \in S(\mathcal{A}).\rho^{\otimes n}(a). \tag{3.115}$$

Note that ϵ_1 is just the evaluation map

$$\begin{aligned} \mathrm{ev}\colon \mathcal{A} &\to CS(\mathcal{A}) \\ a &\mapsto \lambda\rho \in S(\mathcal{A}).\rho(a). \end{aligned} \tag{3.116}$$

Equivalently on morphisms we may define the diagrams as follows. Let $m \leq n$ and $\tau \in \mathbf{Inj}(m,n)$ be an injection $\{1,\dots,m\} \hookrightarrow \{1,\dots,n\}$. We define

$$\eta_\tau\colon \underbrace{\mathcal{A}\otimes\cdots\otimes\mathcal{A}}_{m\text{ times}} \to \underbrace{\mathcal{A}\otimes\cdots\otimes\mathcal{A}}_{n\text{ times}} \tag{3.117}$$

on the algebraic basis by taking $a_1\otimes\cdots\otimes a_m$ to the element $\alpha_1\otimes\cdots\otimes\alpha_n$ which has

$$\alpha_j = \begin{cases} a_i & \text{if } j=\tau(i) \\ 1_{\mathcal{A}} & \text{otherwise.} \end{cases} \tag{3.118}$$

The proof of theorem 3.72 requires that we show a very similar result in the larger category $\mathbf{CSt}_{\mathrm{PU}}$. Before we approach this, it is worth discussing in exactly what way theorem 3.72 deserves the title of *a categorical de Finetti theorem.*

Indeed, the quantum de Finetti theorems of section 3.4 already show the isomorphisms $\mathcal{R}(S(\mathcal{A})) \cong I(\mathcal{A})$ in **ConvCH** for both the minimal and maximal tensor products. The proofs below then only need to show that the space of exchangeable states $I(\mathcal{A})$ is indeed a limit of an appropriate diagram in **ConvCH**, and then the shift to measures on $S(\mathcal{A})$ is taken for free with the isomorphisms above. The diagram in **ConvCH** contains the diagram for which $S\big(\mathcal{A}^{\otimes\mathbb{N}}\big)$ is the limit and so picks out $I(\mathcal{A})$ as a subspace of $S\big(\mathcal{A}^{\otimes\mathbb{N}}\big)$ without much trouble.

Nonetheless, the result itself very much is in the tradition of those original de Finetti theorems and widens their context. In the manner discussed in section 3.5.1, the existence of the colimit above is a statement classifying *parametrised* exchangeable sequence of states on $\mathcal{A}$. As hoped, this classification shows that a parameterised exchange sequence is uniquely defined by a parameterised measure on a single state, just as the non-categorical de Finetti theorems show an unparametrised exchangeable sequence is uniquely defined by a measure on single state. This is clearest once the colimit above is turned into a limit in **ConvCH** via the state-space functor S, or in the latter discussed classical case of theorem 4.16 (parameterised exchangeable measures are a topic of independent study [e.g. 13, 137]), however choosing to work in $\mathbf{CSt}^{\mathrm{op}}_{\mathrm{CPU}}$ instead of **ConvCH** restricts to those spaces and morphisms interesting in quantum foundations.

We now return to showing the case in $\mathbf{CSt}_{\mathrm{PU}}$, the category of C*-algebras and all positive, unital maps.

Theorem 3.73 (Categorical Quantum de Finetti Theorem For Positive Maps)**.** *Let $\mathcal{D}$ be as in theorem 3.72 above for a choice of tensor product, but now with codomain $\mathbf{CSt}_{\mathrm{PU}}$. The colimit of $\mathcal{D}$ in $\mathbf{CSt}_{\mathrm{PU}}$ is $\{\epsilon_n \colon \mathcal{A}^{\otimes n} \to CS(\mathcal{A})\}_{n\in\mathbb{N}}$ with the appropriate tensor product in the domain, for*

$$\epsilon_n(a) = \lambda\rho \in S(\mathcal{A}).\rho^{\otimes n}(a). \tag{3.119}$$

Proof. In what follows, let $\mathcal{A}^{\otimes n} \in \left\{\mathcal{A}^{\otimes n}_{\mathrm{min}}, \mathcal{A}^{\otimes n}_{\mathrm{max}}\right\}$ with $\mathcal{D} \in \{\mathcal{D}_{\mathrm{min}}, \mathcal{D}_{\mathrm{max}}\}$ and $I(\mathcal{A}) \in \{I_{\mathrm{min}}(\mathcal{A}), I_{\mathrm{max}}(\mathcal{A})\}$ the corresponding diagram and space of exchangeable states (though, of course, $I_{\mathrm{max}}(\mathcal{A}) \cong I_{\mathrm{min}}(\mathcal{A})$).

Consider the diagram $\mathcal{D} = \{\eta_\tau \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes m}\}$ of C*-algebras composed with the state-space functor, $S \circ \mathcal{D} = \{S(\eta_\tau) \colon S(\mathcal{A}^{\otimes m}) \to S(\mathcal{A}^{\otimes n})\}$ in **ConvCH**. Since any $\rho \in I(\mathcal{A})$ is exchangeable, its local restrictions $\rho_n = \rho \circ \iota_n$, which are trivially consistent, are also symmetric. In other words, the maps $- \circ \iota_n \colon I(\mathcal{A}) \to S(\mathcal{A}^{\otimes n})$ form a cone for $S \circ \mathcal{D}$.

$$\begin{tikzcd}
S(\mathcal{A}) \arrow[loop above] & S(\mathcal{A}^{\otimes 2}) \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above, out=60, in=30] & S(\mathcal{A}^{\otimes 3}) \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop above] \arrow[loop above, out=150, in=120] \arrow[loop above, out=120, in=90] \arrow[loop above, out=60, in=30] \arrow[loop above, out=90, in=60] & \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \cdots & \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & S(\mathcal{A}^{\otimes n}) \arrow[l, phantom, ""] \arrow[loop above] \arrow[loop above, out=150, in=120] \arrow[loop above, out=120, in=90] \arrow[loop above, out=60, in=30] \arrow[loop above, out=90, in=60] & \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, phantom, "\vdots"] & \cdots \\
& & & & I(\mathcal{A}). \arrow[ullll, bend left] \arrow[ulll, bend left] \arrow[ull, bend left] \arrow[urr, bend right, "-\circ\iota_n"']
\end{tikzcd} \tag{3.120}$$

Claim. *This cone with apex $I(\mathcal{A})$ is the limit of $S \circ \mathcal{D}$ in* **ConvCH***.*

Any collection of states $\{\rho_n \in S(\mathcal{A}^{\otimes n})\}_{n\in\mathbb{N}}$ commuting with the **Inj**-shaped diagram $S \circ \mathcal{D}$ (i.e. any $\{*\}$-apexed cone of the diagram) also commutes with the diagram of the categorical quantum Kolmogorov extension theorem, diag. (3.113). As such, there is a unique state $\rho \in S\left(\mathcal{A}^{\otimes\mathbb{N}}\right)$ such that $\rho_n = \rho \circ \iota_n$ for all $n \in \mathbb{N}$.

Since for any permutation $\sigma \in \mathcal{S}_n$, $\rho_n \circ \eta_\sigma = \rho_n$, ρ is exchangeable and $\rho \in I(\mathcal{A})$. Thus, $I(\mathcal{A})$ is a pointwise limit in **ConvCH**, and thus a limit of the diagram $S \circ \mathcal{D}$, via lemma 3.69.

From theorems 3.65 and 3.66, we have isomorphisms in **ConvCH**

$$I_{\min}(\mathcal{A}) \cong \mathcal{R}(S(\mathcal{A})) \cong I_{\max}(\mathcal{A}) \tag{3.121}$$

and, from theorem 3.26, the further isomorphism $\mathcal{R}(X) \cong SC(X)$.

Thus, the limit of the diagram $S \circ \mathcal{D}$ is $I(\mathcal{A}) \cong SCS(\mathcal{A})$ and since S reflects limits by theorem 3.68, the diagram $\mathcal{D}$ in $\mathbf{CSt}_{\mathrm{PU}}$ has colimit $CS(\mathcal{A})$.

The morphisms of the colimit are the preimages of the maps

$$S(C(S(\mathcal{A}))) \xrightarrow{\cong} \mathcal{R}(S(\mathcal{A})) \xrightarrow{\cong} I(\mathcal{A}) \xrightarrow{-\circ\iota_n} S(\mathcal{A}^{\otimes n}) \tag{3.122}$$

under the state-space functor. To deduce what these maps are, note that no matter the tensor product, the isomorphism $S(\mathcal{A}) \cong \partial I(\mathcal{A})$ is given by $\rho \leftrightarrow \rho^{\otimes\mathbb{N}}$. From this fact, Størmer invokes a previous paper [See 147, Proof of Thm. 3.9] where it is shown that the composition above factors as such:

$$\begin{tikzcd}
S(C(S(\mathcal{A}))) \arrow[r, "\cong"] \arrow[rrd, bend right, "-\circ\epsilon_{\mathbb{N}}"'] & \mathcal{R}(S(\mathcal{A})) \arrow[r, "\cong"] & I(\mathcal{A}) \arrow[r, "-\circ\iota_n"] \arrow[d, hook] & S(\mathcal{A}^{\otimes n}) \\
 & & S\left(\mathcal{A}^{\otimes\mathbb{N}}\right) \arrow[ru, bend right, "-\circ\iota_n"'] &
\end{tikzcd} \tag{3.123}$$

where $\epsilon_{\mathbb{N}} \colon \mathcal{A}^{\otimes\mathbb{N}} \to CS(\mathcal{A})$ is the positive, unital map, $\epsilon_{\mathbb{N}}(b) = \lambda\rho \in S(\mathcal{A}).\rho^{\otimes\mathbb{N}}(b)$.

As such, the limit maps $S(C(S(\mathcal{A}))) \to S(\mathcal{A}^{\otimes n})$ are $- \circ \epsilon_{\mathbb{N}} \circ \iota_n$, and the corresponding $\epsilon_n \colon \mathcal{A}^{\otimes n} \to CS(\mathcal{A})$ in $\mathbf{CSt}_{\mathrm{PU}}$ is just $\epsilon_{\mathbb{N}} \circ \iota_n$, for which

$$\epsilon_n(a)(\rho) = \epsilon_{\mathbb{N}} \circ \iota_n(a)(\rho) = \rho^{\otimes\mathbb{N}}\left(\iota_n(a)\right) = \rho^{\otimes n}(a). \tag{3.124}$$

□

This very naturally transfers to $\mathbf{CSt}_{\mathrm{CPU}}$, where we only have completely positive, unital maps for theorem 3.72:

Proof of theorem 3.72. Since all the maps $\{\eta_\tau\}$ in both diagrams of theorem 3.73 are *-homomorphisms and thus in particular completely positive, the diagrams factor through $\mathbf{CSt}_{\mathrm{CPU}}$.

Since $CS(\mathcal{A})$ is commutative, the colimiting positive maps are also completely positive, as is any mediating map constructed as the colimit in $\mathbf{CSt}_{\mathrm{PU}}$. Thus, $CS(\mathcal{A})$ is also the colimit in $\mathbf{CSt}_{\mathrm{CPU}}$. □

This proof concludes this chapter. The categorical quantum de Finetti theorem of theorem 3.72 with be of great value in what follows and in chapter 4 will be specialised to the commutative case to assist in the study of categorical classical de Finetti theorems.

4

Classical de Finetti Theorems, Multisets, and Symmetric C*-Algebra Tensors

Contents

4.1 Introduction

This chapter proves three de Finetti limit theorems. The first two are classical de Finetti theorems. They classify the space of measures on a compact Hausdorff space X as the limit of diagrams in the Kleisli category of the Radon monad, generalising a theorem of Hewitt and Savage. The first theorem, theorem 4.3, is the subject of section 4.2 and uses a diagram of the product powers X^n of X with permutations and projections. The second, theorem 4.16, explored in section 4.3, follows the approach of Jacobs and Staton, establishing the existence of spaces of multisets $\mathcal{M}_n(X)$ in the category of compact Hausdorff spaces, **CH**, to implicitly encode exchangeability of measures into a simpler diagram in $\mathcal{K}l(\mathcal{R})$.

The final de Finetti theorem of this chapter returns to the quantum setting. Inspired by the classical approach, section 4.4 explores algebras of symmetric tensors as non-commutative duals to spaces of multisets. The main result here is theorem 4.22, a categorical quantum de Finetti theorem again classifying $CS(\mathcal{A})$ as a colimit in $\mathbf{CSt}_{\mathrm{CPU}}$, but of a simpler diagram of spaces of symmetric $\mathcal{A}$-tensors.

4.1.1 Further Background on Classical de Finetti Theorems

In section 3.1, we introduced de Finetti theorems and their use in the context of C*-algebras. This chapter deals with de Finetti theorems for classical probability, where de Finetti himself originally situated the work of *Foresight*. Just as in the quantum case, these theorems classify exchangeable sequences of probabilistic objects, where the probabilistic objects in question here are either random variables or probability measures on measurable spaces. Since the former are special cases of the latter (a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is exchangeable if and only if the associated sequence of laws, that is, the probability distributions defined by $\mathbb{P}\left(X_1 = x_1, \ldots, X_n = x_n\right)$, is exchangeable), in this chapter we elaborate on the case for measures.

De Finetti's theorem classified exchangeable sequences of random variables. The classification was philosophically motivated: *Foresight* is primarily concerned with the justification and elaboration of the subjectivist foundations for probability. He introduces probability initially as a qualitative set of judgements about the comparative likelihoods of events. From there, he explores how to move from these to quantitative descriptions, and how one might use them. Throughout, he maintains that subjective judgement is inherent to the study of probability. To de Finetti, ‘wrong’ probabilistic judgements were only those that were incoherent, for which there existed a guaranteed betting strategy against the implied odds. He showed that a judgement was exploitable in this way if and only if it violated what we would now consider the axioms of probability: the total probability of exhaustive events is 1; the probability of the union of disjoint events is the sum of their probabilities. On the other hand, unlike, for example, frequentist probability theorists who grounded the study of probability in the assertion of ‘true’ probabilities which the work of statistics would be attempting to approximate, de Finetti felt that no such truth existed, and studying probability was only about understanding what judgements entailed each other.

This approach precluded the use of constructs that are now used without concern throughout probability and statistics. His theorem, which provides the foundation of this thesis, is an attempt to recover something of the power of independent and identically distributed random variables. Within subjectivist philosophy, where a distribution is a judgement of comparative likelihood, the existence of events whose probability distributions are judged to follow an i.i.d. law is nonsensical, or, at least, a vastly heavy assumption. There is no meaning in this formalism for a ‘hidden’ distribution which generates a sequence of independent instantiations.

Here de Finetti identifies a dire problem, since without i.i.d. sequences, it is not clear from where inference arises. Suppose Gilda is going to start flipping a coin. The coin is of irregular appearance, but, based on her experiences of coins, and some belief that this coin does not seem that different to previous coins, she judges that every flip of the coin is equally likely to show heads or tails. This is a valid judgement and is certainly is not inconsistent. Upon beginning to flip the coin, though, she begins to get suspicious: the first seven flips are all heads, then a couple of tails, more heads. By one hundred flips (and a long 15 minutes), eighty-five of the flips were heads. If she had had this information before beginning the experiment, she likely would have made a different judgement about the likelihood of heads on the one-hundred-and-first throw. Why, if judgements are subjective and cannot be wrong? And how does the subjectivist reconcile this for future flips? Is Gilda doomed to continue to insist on even odds even when, ten months later, a million throws has shown heads eighty percent of the time?

The frequentist solves inference by supposing all flips are independently and identically distributed $X_n \sim \mathrm{Ber}(\mu)$ according to some unknown 'true' bias towards heads of $\mu \in [0,1]$ and then approximates μ via the law of large numbers:

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \{X_i = \text{Heads}\} \quad \text{where } \{P\} = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

To the frequentist, the initial judgement of likelihood is completely irrelevant to the mathematical statement.

The objective Bayesian pulls a similar trick, though allows at least a formal statement of the belief in the form of a prior distribution over biases on $[0,1]$ and then updates their belief with observed results via Bayes rule. All flips are still considered to be independently and identically distributed $X_n \sim \mathrm{Ber}(\mu)$ according to some unknown 'true' bias towards heads of $\mu \in [0,1]$.

The subjectivist probability theorist is left without either of these possible angles, since the existence of μ is impossible to express as the judgement of some observer. De Finetti resolves to solve the problem of inference and does so with exchangeability. Where other solutions to the inference problem theorise hidden true distributions, de Finetti suggests that we expect to be able to do inference under subjective probabilities because we have already made the assumption of exchangeability. Exchangeability is a judgement of its own about the indistinguishability of trials: though I do not have philosophical justification to say that every flip of a coin is a probabilistic repetition of some identical event, I can quite straightforwardly assert that I, for one, don't really know the difference between them, and thus judge different orders of the same results to be equally likely.

De Finetti's theorem says that under this assumption we do not get back pure i.i.d. sequences, but we do get the next best thing: an exchangeable sequence of random variables is equivalent to *a mixture of* i.i.d. sequences. Explicitly, if X_n is an exchangeable sequence of random variables for which one has made an exchangeable judgement about likelihoods, which is to say one may ask '*What* *is the likelihood that* $X_{j_1} = x_1, X_{j_2} = x_2, \ldots, X_{j_n} = x_n$*?'* and the answer will be the same under any permutation of the indices $\{j_1, \ldots, j_n\}$, then there exists a distribution Φ on random variables (up to equality in distribution) such that the value $\mathbb{P}\left(X_{j_1} = x_1, X_{j_2} = x_2, \ldots, X_{j_n} = x_n\right)$ is equal to the expected value of $\prod_{i=1}^{n} \mathbb{P}\left(Y = x_i\right)$ when drawing a random variable Y using Φ. In other words, this is the likelihood of the getting the sequence of results $(x_1, \ldots, x_n)$ by choosing a random variable Y at random using Φ, and then forming identical and independent repetitions of Y.

To formulate this idea properly, the back and forth between measures and functionals of theorem 3.26 and the use of the Radon monad is essential. In what follows, recall from definition 2.28 that $f\colon X \rightsquigarrow Y$ denotes a morphism in the Kleisli category of a probability monad. Such an f can be seen as a measure on Y parametrised by elements of X. This notation is also used in diagrams, where composition is understood as happening in the Kleisli category.

The process of generating a probability on sequences $(x_1, \ldots, x_n)$ by choosing a Y and then replicating it identically and independently is more easily understood as the composition of probabilistic morphisms, or morphisms in the Kleisli category of a probability monad: informally, a distribution on random variables is a Kleisli morphism $\Phi\colon \{*\} \rightsquigarrow \mathrm{RV}_\Omega$, whilst $\mathrm{iid}_n\colon \mathrm{RV}_\Omega \rightsquigarrow \mathbb{R}^n$ is a morphism which takes a random variable to the i.i.d. law it defines on $\mathbb{R}^n$. The theorem described above says there exists a unique measure such that the following diagram commutes for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
& \mathrm{RV}_\Omega \arrow[dd, squiggly, "\mathrm{iid}_n"] \\
\{*\} \arrow[ur, dashed, squiggly, "\exists\Phi"] \arrow[dr, squiggly, "\mathbb{P}(X_1=-,X_2=-,\ldots,X_n=-)"'] & \\
& \mathbb{R}^n
\end{tikzcd} \tag{4.2}$$

This should look familiar.

A number of extensions have been explored beyond de Finetti's work. Other forms of symmetries provide similar or even equivalent formulation, for example stationary sequences of real numbers are equivalent to exchangeable ones and so de Finetti's theorem may be applied to them [135], or the theorem as a special case of ergodic decomposition [158]. In the vein of the work of this chapter, extensions of de Finetti's theorem to certain parametrised circumstances (namely to measures symmetric around a location parameter) were explored by Diaconis and Freedman [44], and there have been specific recent applications to parameterised Pólya urns [13, 137]. Computability of the de Finetti measure has also been studied [45]. In purely extending the types of measurable spaces on which exchangeable sequences of measures could be classified, the most general result is that of Hewitt and Savage which extends de Finetti's theorem to compact Hausdorff spaces with their Borel σ-algebras.

De Finetti's focus on finitely-supported random variables was philosophically informed but was also used in the proof and as such in generalising the theorem thought was given to the number of ways of moving from the finite to the infinite in probability. One may consider probability spaces with countable underlying sets, or σ-finite measures. Alternatively, finite measurable spaces may also be understood as arising from the discrete topology on finite sets, and as such one might take a topological route of generalising discrete finiteness, and consider compactness.

Compactness describes a certain kind of 'smallness' for a topological space. For such a space, there is no 'creeping away to infinity', no hiding tricky properties in the further reaches of a space. As such we get the extreme value theorem: if $f\colon X \to \mathbb{R}$ is a continuous function on a compact space X, then f is bounded and attains its bounds. If we are interested in discrete finite measurable spaces exactly because of their 'containedness', the compact spaces share many such properties.

Alternatively, we may be interested in discrete finite measurable spaces because of the discreteness, the fact that points are easy to distinguish. In this case, the property we are looking for might be that of being Hausdorff. Compact topological spaces restrict escape to infinity by not having too fine a topology; on the other hand, Hausdorff spaces are fine enough to separate points. Together, compactness and the Hausdorff condition make very strong demands on the structure of a space, and it is this well-behavedness that is summarised in the properties of monadicity of $\mathbf{CH} \to \mathbf{Set}$ (theorem 2.16) and the isomorphism $\mathbf{CH} \cong \mathbf{cCSt}^{\mathrm{op}}_{\mathrm{MIU}}$ (theorem 3.10).

The endgame of this particular line of inquiry is the de Finetti theorem of Hewitt and Savage.

Theorem 4.1 (Hewitt-Savage-de Finetti Theorem [75, Thm.7.2])**.** *Let X be a compact Hausdorff space. Let $\mu \in \mathcal{R}\big(X^{\mathbb{N}}\big)$ be a measure such that for each $n \in \mathbb{N}$, the pushforward of μ by the projection $p_n\colon X^{\mathbb{N}} \to X^n$, $\mu_n := (p_n)_*\mu \in \mathcal{R}(X^n)$ is symmetric.*

Then there exists a Radon measure Φ on $\mathcal{R}(X)$ such that

$$\mu(A) = \int_{\nu \in \mathcal{R}(X)} \nu^{\mathbb{N}}(A)\,\mathrm{d}\Phi. \tag{4.3}$$

Note that this means $\Phi \in \mathcal{R}^2(X) := \mathcal{R}(\mathcal{R}(X))$.

Recall [e.g. 65, P.157, Thm. B] that the countably finite product measure $\nu^{\mathbb{N}} \in \mathcal{R}\big(X^{\mathbb{N}}\big)$ is the unique measure with the property that, for all $n \in \mathbb{N}$ and measurable $A_i \subset X$ for $i = 1, \dots, n$, $\nu^{\mathbb{N}}\left(A_1 \times \cdots \times A_n \times X^{\mathbb{N}}\right) = \prod_{i=1}^{n} \nu\left(A_i\right)$. In particular then, we have that for all $n \in \mathbb{N}$

$$\mu_n(A) = \int_{\nu \in \mathcal{R}(X)} \nu^n(A)\,\mathrm{d}\Phi. \tag{4.4}$$

This is a necessary and sufficient equation to define μ, since μ is uniquely determined by the sequence μ_n (this is the Kolmogorov extension theorem, theorem 2.30).

As in diag. (4.2), this Φ and the property described in eq. (4.4) is actually representable by the commutativity of a diagram in the Kleisli category of the Radon monad:

$$\begin{tikzcd}
 & \mathcal{R}(X) \arrow[dd, squiggly, "\mathrm{iid}_n"] \\
\{*\} \arrow[ur, dashed, "\exists\Phi"] \arrow[dr, squiggly, "\mu_n"'] & \\
 & X^n
\end{tikzcd} \tag{4.5}$$

where $\mathrm{iid}_n\colon \mathcal{R}(X) \rightsquigarrow X^n$ now takes a measure $\mu \in \mathcal{R}(X)$ to the n-times product measure μ^n above.

This is the seed of a categorical treatment of this theorem. Just as how exchangeable states of a C*-algebra may be represented as a cone of a diagram, we may represent sequences of measures as cones of diagrams. For example, given a compact Hausdorff space X, we may consider the $(\mathbb{N}, \leq)^{\mathrm{op}}$-indexed diagram in **CH** of $\mathcal{R}(X^n)$ for all $n \in \mathbb{N}$ and with maps $\mathcal{R}(X^{n+1}) \to \mathcal{R}(X^n)$ given by $\mathcal{R}(\pi)$, for the projection $\pi\colon (x_i)_{i=1}^{n+1} \mapsto (x_i)_{i=1}^{n}$ dropping the last component. A cone of this diagram with apex $\{*\}$ is just a choice of measure $\mu_n \in \mathcal{R}(X^n)$ for each n, that are consistent in the sense that $\mu_n = \pi_* \mu_{n+1}$.

$$\begin{tikzcd}
\mathcal{R}(X) & \mathcal{R}(X^2) \arrow[l] & \dots \arrow[l] & \mathcal{R}(X^n) \arrow[l] & \arrow[l] \\
& & & \{*\} \arrow[lllu, bend left] \arrow[llu, bend left] \arrow[u] & .
\end{tikzcd} \tag{4.6}$$

Since the codomain of all the maps in this diagram are of the form $\mathcal{R}(Y)$ for some $Y \in \mathbf{CH}$, we may ask if this diagram can be understood in the Kleisli category of $\mathcal{R}$, which it can. The composition

$$\{*\} \xrightarrow{\mu_{n+1}} \mathcal{R}(X^{n+1}) \xrightarrow{\mathcal{R}(\pi)} \mathcal{R}(X^n) \tag{4.7}$$

is exactly the Kleisli composition

$$\{*\} \overset{\mu_{n+1}}{\rightsquigarrow} X^{n+1} \overset{\mathcal{R}(\pi)\circ\eta}{\rightsquigarrow} X^n. \tag{4.8}$$

As such, a consistent sequence of measures as above is equivalently described as a $\{*\}$-cone over the $(\mathbb{N}, \leq)^{\mathrm{op}}$-diagram in $\mathcal{K}l(\mathcal{R})$ of X^ns and the maps $X^{n+1} \rightsquigarrow X^n$ given by deterministic projections π:

$$\begin{tikzcd}
X & X^2 \arrow[l] & \dots \arrow[l] & X^n \arrow[l] & \arrow[l] \\
& & & \{*\} . \arrow[lllu, squiggly, bend left] \arrow[llu, squiggly, bend left] \arrow[u, squiggly] &
\end{tikzcd} \tag{4.9}$$

Recall a morphism $X \rightsquigarrow Y$ is called *deterministic* if it of the form $\eta_Y \circ f$ for some $f\colon X \to Y$ in **CH**. We will often write f in diagrams with a regular arrow instead of zigzag arrows, when really calculations will be done with $\eta_Y \circ f$.

Exchangeability of such a sequence of measures μ_n is invariance under pushforward by the braiding homeomorphisms $X^\sigma\colon X^n \xrightarrow{\sim} X^n$ for $\sigma \in \mathcal{S}_n$ which permute the factors of the product. Adding these deterministic maps to diag. (4.9), we can see an exchangeable sequence of measures as a $\{*\}$-cone over an $\mathbf{Inj}^{\mathrm{op}}$-shaped diagram, like for C*-algebras but with the arrows reversed:

$$\begin{tikzcd}
X \arrow[loop above] & X^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above] & X^3 \arrow[l, shift left=4] \arrow[l, shift left=2] \arrow[l, shift left=0] \arrow[l, shift right=2] \arrow[l, shift right=4] \arrow[l, shift right=6] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & X^n \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \\
& & & \{*\} . \arrow[ulll, rightsquigarrow, bend left] \arrow[ull, rightsquigarrow, bend left] \arrow[ul, rightsquigarrow] \arrow[ur, rightsquigarrow] & &
\end{tikzcd} \tag{4.10}$$

De Finetti's theorem, and the Hewitt-Savage-de Finetti Theorem, then suggest that cones of this form factor uniquely through $\mathcal{R}(X)$. In what follows, it is shown that this is not just true of cones of this form, but any cone over the diagram. In other words, in $\mathcal{K}l(\mathcal{R})$, $\mathcal{R}(X)$ is the limit of this diagram. The limit maps $\mathrm{iid}_n\colon \mathcal{R}(X) \rightsquigarrow X^n$ are exactly those that take a measure $\mu \in \mathcal{R}(X)$ to the product measure $\mu^n \in \mathcal{R}(X^n)$.

$$\begin{tikzcd}
X \arrow[loop above] & X^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above] & X^3 \arrow[l, shift left=4] \arrow[l, shift left=2] \arrow[l, shift left=0] \arrow[l, shift right=2] \arrow[l, shift right=4] \arrow[l, shift right=6] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & X^n \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \arrow[loop above] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \\
& & & & \mathcal{R}(X) \arrow[u, rightsquigarrow, "\mathrm{iid}_n"] \arrow[ullll, rightsquigarrow, bend left] \arrow[ulll, rightsquigarrow, bend left] \arrow[ull, rightsquigarrow, bend left] & \\
& & & & Y. \arrow[u, rightsquigarrow, dashed, "\exists!"] \arrow[uullll, rightsquigarrow, bend left] \arrow[uulll, rightsquigarrow, bend left] \arrow[uull, rightsquigarrow, bend left] \arrow[uu, rightsquigarrow, bend right] &
\end{tikzcd} \tag{4.11}$$

4.1.2 Jacobs and Staton's de Finetti limit

The viability of a categorical limit approach to de Finetti theorems was shown in Jacob and Staton's 2020 paper *De Finetti's Construction as a Categorical Limit* [91]. Inspired by de Finetti's original use of finitely-supported random variables, they study the case that X is the discrete space on two elements $\mathbf{2} := \{0, 1\}$ and do so not in the Kleisli category of the Radon monad, but instead in the Kleisli category of the Giry monad $\mathcal{G}$, which acts on the larger category of **Meas**. The limit object is the same, since $\mathcal{G}(2) \cong [0,1] \cong \mathcal{R}(2)$ as measurable spaces. Where these results do and do not overlap is discussed in section 4.3.4.

4.1.3 Exchangeability and Multisets

Another innovation of Jacobs and Staton [91] is the use of multisets. A multiset (definition 2.21) is a finite, unordered collection of elements of a set A, with repeated elements permitted. They can formally be represented by finitely supported maps $\phi\colon \mathbb{N} \to A$, or by equivalence classes of n-tuples of elements of A. The equivalence relation at play here is about permutations: two n-tuples are equivalent if and only if they are permutations of each other, and in this way multisets encode symmetry. A symmetric measure on X^n intuitively gives rise to a measure on the set of multisets, and from measures on sets of multisets we may build permutation invariant measures on products. Jacobs and Staton use multisets in their de Finetti limit to quotient exchangeability 'inside' the objects of the diagram, and then introduce a permutation-invariant projection operation called Draw-and-Delete, to encode consistency. In this chapter, I explore first the product form of a de Finetti limit, and then show that this also implies a multiset form, where slight care must now be shown to situate the multiset monad in **CH**.

4.2 The Kleisli Category de Finetti Limit

This section deals with the statement of a de Finetti theorem as a limit in a category of probabilistic processes, the Kleisli category of the Radon monad. This is the appropriate choice for the de Finetti theorem of Hewitt and Savage.

4.2.1 The Inclusion $\mathcal{K}l(\mathcal{R}) \hookrightarrow$ ConvCH

The results in this section rely heavily on using the categorical quantum de Finetti theorem for $\mathbf{CSt}_{\mathrm{PU}}$ of theorem 3.73 to construct limits in **ConvCH**, and then to reflect those limit into the full subcategory $\mathcal{K}l(\mathcal{R})$ exploiting the fact that $\mathbf{ConvCH} \cong \mathcal{E}m(\mathcal{R})$.

Here, we explore exactly what the inclusion $\mathcal{K}l(\mathcal{R}) \hookrightarrow \mathbf{ConvCH}$ looks like explicitly.

The proof of the equivalence $\mathbf{ConvCH} \cong \mathcal{E}m(\mathcal{R})$ is first given in [150], whilst [140] allows an explicit statement without monadicity theorems. It is as such: let $\gamma\colon \mathcal{R}(X) \to X \in \mathcal{E}m(\mathcal{R})$ be an algebra of the Radon monad, then there is a homeomorphism in $\mathbf{CH}$, $X \cong U_{\mathbf{ConvCH}}\mathbf{W}$, for some $\mathbf{W} \in \mathbf{ConvCH}$ and the forgetful functor $U_{\mathbf{ConvCH}}\colon \mathbf{ConvCH} \to \mathbf{CH}$, such that as algebras $\gamma\colon \mathcal{R}(X) \to X$ is isomorphic to the algebra $\mathcal{R}(U_{\mathbf{ConvCH}}\mathbf{W}) \to U_{\mathbf{ConvCH}}\mathbf{W}$ which takes a measure to its barycentre: *the barycenter of a measure* $\mu \in \mathcal{R}(\mathbf{W})$ is the unique point $w_\mu \in \mathbf{W}$ with the property that for all affine, continuous functionals $\phi\colon \mathbf{W} \to \mathbb{C}$,

$$\int_{\mathbf{W}} \phi \,\mathrm{d}\mu = \phi(w_\mu). \tag{4.12}$$

All Radon measures on spaces in $\mathbf{ConvCH}$ have a barycentre [128, Prop. 1.1].

The free algebras of $\mathcal{R}$ are the algebras of the form $\mathrm{m}_{\mathcal{R}}\colon \mathcal{R}^2(X) \to \mathcal{R}(X)$ for $X \in \mathbf{CH}$. In this case, the isomorphism above is equality. In other words, the intensity measure $\mathrm{m}_{\mathcal{R}}(\Phi)$ of a random measure $\Phi \in \mathcal{R}^2(X)$ is its barycentre.

To see this, suppose that $\Phi = \sum_i a_i \delta_{\mu_i}$ is a finitely-supported discrete measure, with $\sum_i a_i = 1$ and $\mu_i \in \mathcal{R}(X)$. Then, for a continuous and affine $\phi\colon \mathcal{R}(X) \to \mathbb{C}$,

$$\begin{aligned}\int_{\mathcal{R}(X)} \phi(\mu)\,\mathrm{d}\Phi(\mu) &= \sum_i a_i \phi(\mu_i) \\ &= \phi\left(\sum_i a_i \mu_i\right) \\ &= \phi\left(\mathrm{m}_{\mathcal{R}}(\Phi)\right).\end{aligned} \tag{4.13}$$

For $\Phi \in \mathcal{R}^2(X)$, there exists a sequence $(\nu_j)_{j\in\mathbb{N}} = \left(\sum_{i=1}^{n_j} a_{ij}\delta_{\mu_{ij}}\right)_{j\in\mathbb{N}}$ of discrete measures converging to Φ in $\mathcal{R}^2(X)$ (in fact, we may even choose these so that they have the same barycenter as Φ [106, Thm. 8]). Since both $\mathrm{m}_{\mathcal{R}}$ and ϕ are continuous,

$$\begin{aligned}
\phi \circ \mathrm{m}_{\mathcal{R}}(\Phi) &= \phi \circ \mathrm{m}_{\mathcal{R}}\left(\lim_j \nu_j\right) && (4.14)\\
&= \lim_j \left[\phi \circ \mathrm{m}_{\mathcal{R}}(\nu_j)\right] && (4.15)\\
&= \lim_j \int_{\mathcal{R}(X)} \phi \, \mathrm{d}\nu_j && (4.16)\\
&= \int_{\mathcal{R}(X)} \phi \, \mathrm{d}\left[\lim_j \nu_j\right] && (4.17)\\
&= \int_{\mathcal{R}(X)} \phi \, \mathrm{d}\Phi. && (4.18)
\end{aligned}$$

Equation (4.16) follows since ν_j is discrete. Equation (4.17) is permitted since $\lim_j \mu_j = \mu$ in $\mathcal{R}(Y)$ means exactly that for all $\psi \in C(Y)$ and $\Omega \subset \mathbb{C}$ open such that $\int_Y \phi \, \mathrm{d}\mu \in \Omega$, there exists some $J \in \mathbb{N}$ such that $\int_Y \psi \, \mathrm{d}\mu_j \in \Omega$ for all $j \geq J$ and in particular, given ψ and an arbitrary $\epsilon > 0$, we can choose Ω to show that

$$\left|\int_Y \psi \, \mathrm{d}\mu_j - \int_Y \psi \, \mathrm{d}\mu\right| < \epsilon \tag{4.19}$$

eventually. Thus, for any $\psi \in C(Y)$,

$$\lim_j \int_Y \psi \, \mathrm{d}\mu_j = \int_Y \psi \, \mathrm{d}\mu. \tag{4.20}$$

This shows that the full embedding $\mathcal{K}l(\mathcal{R}) \hookrightarrow$ **ConvCH** takes a space X to $\mathcal{R}(X) \in$ **ConvCH** and on morphisms, $f \colon X \rightsquigarrow Y$ becomes the affine and continuous map $\mathrm{m}_{\mathcal{R}} \circ \mathcal{R}(f) \colon \mathcal{R}(X) \to \mathcal{R}(Y)$.

$$\begin{tikzcd}
 & \mathcal{R}(\mathcal{R}(Y)) \arrow[dr, "\mathrm{m}_{\mathcal{R}}"] & \\
\mathcal{R}(X) \arrow[ur, "\mathcal{R}(f)"] \arrow[rr] & & \mathcal{R}(Y).
\end{tikzcd} \tag{4.21}$$

4.2.2 Categorical Kolmogorov Extension Theorem

The Kolmogorov extension theorem theorem 2.30 says that measures on countable products of certain spaces are defined by their finite marginals. In other words, if two measures on $X^{\mathbb{N}}$ are equal under pushforward by all projections $p_n \colon X^{\mathbb{N}} \to X^n, (x_i)_{i\in\mathbb{N}} \mapsto (x_1, \ldots, x_n)$ for $n \in \mathbb{N}$, then they are the same. It also says that any sequence of measures μ_n on $X^{\mathbb{N}}$ which are compatible in the sense that μ_n is the pushforward of μ_{n+1} by the projection $X^{n+1} \to X$ which drops the final entry, can be stitched together to make a measure μ on $X^{\mathbb{N}}$. The first part ensures that this measure is unique. As dealt with before, this is a statement about diagrams:

$$\begin{tikzcd}
X & X^2 \arrow[l] & \dots \arrow[l] & X^n \arrow[l] & {} \arrow[l] \\
 & & & X^{\mathbb{N}} \arrow[u, "p_n"] \arrow[ulll, bend left] \arrow[ull, bend left] & \\
 & & & \{*\}\,. \arrow[u, dashed, squiggly, "\exists!\mu"] \arrow[uu, squiggly, bend right, "\mu_n"'] \arrow[uulll, squiggly, bend left] \arrow[uull, squiggly, bend left] &
\end{tikzcd} \tag{4.22}$$

Note that zigzag arrows are used for potentially non-deterministic maps, whilst regular arrows are deterministic maps. Composition is always considered probabilistically: as Kleisli composition. Explicitly, given $\phi\colon X \rightsquigarrow Y$ and $g\colon Y \to Z$, the composition $X \rightsquigarrow Z$ as a Kleisli map will be $\mathcal{R}(g) \circ \phi\colon X \to \mathcal{R}(Z)$ in **CH**. So, as is familiar by this point, we may ask: can we replace $\{*\}$ with a general parametrizing space Y?

Since we are situating this theorem in a Kleisli category, a natural choice at this point is to yet again use $\mathcal{K}l(\mathcal{R})$, in which case we are concerned with measures on the n-powers of a compact Hausdorff space X continuously parameterised (the sense of the topology on $\mathcal{R}(X^n)$) by another compact Hausdorff space Y.

$$\begin{tikzcd}
X & X^2 \arrow[l] & \dots \arrow[l] & X^n \arrow[l] & {} \arrow[l] \\
 & & & X^{\mathbb{N}} \arrow[u, "p_n"] \arrow[ulll, bend left] \arrow[ull, bend left] & \\
 & & & Y. \arrow[u, dashed, squiggly, "\exists!\mu(-|y)"] \arrow[uu, squiggly, bend right, "\mu_n(-|y)"'] \arrow[uulll, squiggly, bend left] \arrow[uull, squiggly, bend left] &
\end{tikzcd} \tag{4.23}$$

If all the maps in diag. (4.23) were deterministic, there is a unique deterministic map $Y \to X^{\mathbb{N}}$. This is exactly because of the definition of $X^{\mathbb{N}}$ as a product in **CH**. On the other hand, if they are not, then we may define a map $Y \rightsquigarrow X^{\mathbb{N}}$ by considering each $y \in Y$ individually and constructing the μ of diag. (4.22), as we did in lemma 3.69 for pointwise limits in the category of (cancellative) convex, compact Hausdorff spaces, **ConvCH** (recall definition 2.18). The question then is whether this map is continuous. It is.

Theorem 4.2 (Categorical Classical Kolmogorov Extension Theorem)**.** *Let* $X \in$ **CH**. *Let* $\pi_{n+1}\colon X^{n+1} \to X^n$ *be the projection* $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$. *The* $(\mathbb{N}, \leq)^{\mathrm{op}}$*-shaped (deterministic) diagram in* $\mathcal{K}\ell(\mathcal{R})$ *of* X^n*s and the projections* π_{n+1} *has limit* $X^{\mathbb{N}}$ *with arrows* $p_n\colon X^{\mathbb{N}} \to X^n$.

Proof. This is a specialisation of the quantum case. Recall theorem 3.71, the categorical quantum Kolmogorov extension theorem: for a C*-algebra $\mathcal{A}$, the limit of the $(\mathbb{N}, \leq)^{\mathrm{op}}$-shaped diagram of state spaces in **ConvCH**

$$\begin{tikzcd} S(\mathcal{A}) & S(\mathcal{A}^{\otimes 2}) \arrow[l, "-\circ\iota_{12}"'] & \dots \arrow[l, "-\circ\iota_{23}"'] & S(\mathcal{A}^{\otimes n}) \arrow[l, "-\circ\iota_{(n-1)n}"'] & \dots \arrow[l, "-\circ\iota_{n(n+1)}"'] \end{tikzcd} \tag{4.24}$$

is $S\big(\mathcal{A}^{\otimes\mathbb{N}}\big)$ with the inclusions $- \circ \iota_n\colon S\big(\mathcal{A}^{\otimes\mathbb{N}}\big) \to S(\mathcal{A}^{\otimes n})$.

Let $\mathcal{A} = C(X)$. Note in addition from proposition 3.45 that $\mathcal{A}^{\otimes n} = C(X)^{\otimes n} \cong C(X^n)$. As such the objects of the above diagrams become the spaces $SC(X^n) \cong \mathcal{R}(X^n)$. The morphisms follow two diagram chases. Firstly we consider the clockwise morphisms in $\mathbf{CSt}_{\mathrm{CPU}}$:

$$\begin{tikzcd} C(X^n) \arrow[r, "\sim"] \arrow[d, dashed] & C(X)^{\otimes n} \arrow[d, "\iota_{n(n+1)}"] \\ C(X^{n+1}) & C(X)^{\otimes n+1} \arrow[l, "\sim"] \end{tikzcd} \tag{4.25}$$

For $\phi \in C(X^n)$, the top isomorphism takes this to an element $\sum_{j=1}^m \phi_{1j} \otimes \cdots \otimes \phi_{nj} \in C(X)^{\otimes n}$ for some $m \in \mathbb{N}$ and $\phi_{ij} \in C(X)$ such that $\phi(x_1, \dots, x_n) = \sum_{j=1}^m \prod_{i=1}^n \phi_{ij}(x_i)$. Then the vertical arrow takes this to $\sum_{j=1}^m \phi_{1j} \otimes \cdots \otimes \phi_{nj} \otimes 1$ where 1 is the constant function $X \to \mathbb{C}$. Finally, the bottom isomorphism takes this to the function $\phi'(x_1, \dots, x_n, x_{n+1}) = \sum_{j=1}^m \left[\prod_{i=1}^n \phi_{ij}(x_i) \times 1\right] = \sum_{j=1}^m \prod_{i=1}^n \phi_{ij}(x_i) = \phi(x_1, \dots, x_n)$ which is to say that $\phi' = \phi \circ p_n$ and the downwards arrow is $C(p_n) = - \circ p_n$.

The second diagram chase is the anti-clockwise direction of this diagram

$$\begin{tikzcd} \mathcal{R}(X^n) & SC(X^n) \arrow[l, "\sim"'] \\ \mathcal{R}(X^{n+1}) \arrow[u, dashed] \arrow[r, "\sim"'] & SC(X^{n+1}) \arrow[u, "-\circ(-\circ p_n)"'] \end{tikzcd} \tag{4.26}$$

A measure $\mu_{n+1} \in \mathcal{R}(X^{n+1})$ in the bottom left becomes the state $\int_{X^{n+1}} - \, \mathrm{d}\mu_{n+1}$ on the bottom right, and this translates to

$$\int_{X^{n+1}} (- \circ p_n) \, \mathrm{d}\mu_{n+1} = \int_{X^{n+1}} - \, \mathrm{d} \, (p_n)_* \, \mu_{n+1} \tag{4.27}$$

in $SC(X^n)$. Thus, the appropriate measure in $\mathcal{R}(X^n)$ is $(p_n)_* \, \mu_{n+1}$ and the vertical morphism we are looking for is the probabilistic lift of the deterministic map p_n.

Thus, diag. (4.24) becomes

$$\mathcal{R}(X) \xleftarrow{(p_2)_*} \mathcal{R}(X^2) \xleftarrow{(p_3)_*} \dots \xleftarrow{(p_n)_*} \mathcal{R}(X^n) \xleftarrow{(p_{n+1})_*} \dots . \tag{4.28}$$

Applying the quantum categorical Kolmogorov extension theorem here says that the limit of such a diagram in **ConvCH** is $S\Big(C(X)^{\otimes \mathbb{N}}\Big)$.

It is a common result of C*-algebras, see for example Takeda in 1955 [151, Thm. 2 and Cor.], that inductive limits (colimits over directed sets) of commutative C*-algebras translate over the equivalence C to projective limits (limits over directed sets) of the corresponding compact Hausdorff spaces. Thus,

$$\begin{aligned} C(X)^{\otimes \mathbb{N}} &= \operatorname*{colim}_{\mathbf{CSt}_{\mathrm{MIU}}} \Big[\dots \to C(X)^{\otimes n} \to \dots \Big] \\ &= \operatorname*{colim}_{\mathbf{CSt}_{\mathrm{MIU}}} \left[\dots \to C(X^n) \to \dots \right] \\ &= C\Big(\lim_{\mathbf{CH}} \left[\dots \leftarrow X^n \leftarrow \dots \right] \Big) \\ &= C\Big(X^{\mathbb{N}}\Big). \end{aligned} \tag{4.29}$$

Thus, the limit is $SC\Big(X^{\mathbb{N}}\Big) \cong \mathcal{R}\Big(X^{\mathbb{N}}\Big)$, with morphisms $\mathcal{R}(p_n) \colon X^{\mathbb{N}} \to X^n$ in **ConvCH**. This limit cone over diag. (4.28) is just the image of $p_n \colon X^{\mathbb{N}} \to X^n$ as a deterministic cone in $\mathcal{K}\ell(\mathcal{R})$ over the diagram of the theorem, of the projections $\pi_{n+1} \colon X^{n+1} \to X^n$ under the inclusion $\mathcal{K}\ell(\mathcal{R}) \to$ **ConvCH**. This inclusion, being full and faithful, reflects limits, and we are done. □

Theorem 4.2 realises $X^{\mathbb{N}}$ as a *Kolmogorov product*, in the Markov category $\mathcal{K}\ell(\mathcal{R})$. We have been unable to find a statement of this result in the literature (Jacobs and Staton only mention the case for **2** in $\mathcal{K}\ell(\mathcal{G})$ in passing and literature on Kolmogorov products has not yet shown which do or do not exist in $\mathcal{K}\ell(\mathcal{R})$).

It is worth noting that Kozen has also approached the Kolmogorov extension theorem via a similar method, making progress towards a limit theorem using Radon spaces (spaces on which all measures are Radon) and reversible Markov kernels [112].

4.2.3 Hewitt-Savage-de Finetti theorem as a Categorical Limit

For $\tau \in \mathbf{Inj}(m,n)$, define $X^\tau \colon X^n \to X^m$ by

$$X^\tau(x_1, \dots, x_n) = \Big(x_{\tau(1)}, \dots, x_{\tau(m)}\Big) . \tag{4.30}$$

Note that if $\sigma \in \mathbf{Inj}(n,n) = \mathcal{S}_n$, X^σ agrees with the braiding map of the same notation.

Theorem 4.3 (Categorical Hewitt-Savage de Finetti Theorem in $\mathcal{K}l(\mathcal{R})$)**.** *Let X be a compact Hausdorff space. Consider the deterministic diagram $X^- \colon \mathbf{Inj}^{\mathrm{op}} \to \mathcal{K}l(\mathcal{R})$ which takes n to X^n and, for natural numbers $m \leq n$, an injective function $\tau \in \mathbf{Inj}(m,n)$ to the map $X^\tau \colon X^n \to X^m$.*

The limit of this diagram is the space of measures on X, $\mathcal{R}(X)$, with limit projections $\mathrm{iid}_n \colon \mathcal{R}(X) \rightsquigarrow X^n$ taking a measure $\mu \in \mathcal{R}(X)$ to the product measure $\mu^n \in \mathcal{R}(X^n)$.

This result follows from an intermezzo version in **ConvCH**.

Theorem 4.4 (Categorical Hewitt-Savage-de Finetti Theorem in **ConvCH**)**.** *Let X be a compact Hausdorff space. Consider the diagram $\mathbf{Inj}^{\mathrm{op}} \to \mathbf{ConvCH}$ which takes n to $\mathcal{R}(X^n)$ and, for natural numbers $m \leq n$, an injective function $\tau \in \mathbf{Inj}(m,n)$ to the map $\mathcal{R}(X^\tau) = (X^\tau)_* \colon \mathcal{R}(X^n) \to \mathcal{R}(X^m)$.*

The limit of this diagram is $\mathcal{R}^2(X)$, with limit projections $\mathcal{R}^2(X) \to \mathcal{R}(X^n)$ taking a measure $\Phi \in \mathcal{R}^2(X)$ to the measure

$$\int_{\mu \in \mathcal{R}(X)} \mu^n(-) \, \mathrm{d}\Phi. \tag{4.31}$$

Here, $\mathcal{R}^2(X)$ is simply the application of the functor twice: $\mathcal{R}(\mathcal{R}(X))$. Its elements are measures on the space of measures on X.

Proof. Recall theorem 3.73: Let $\mathcal{A}$ be a C*-algebra. Choose either the maximal or minimal C*-tensor product. The diagram $\mathcal{D}\colon \mathbf{Inj} \to \mathbf{CSt}_{\mathrm{PU}}$ takes n to $\mathcal{A}^{\otimes n}$ and an injection $\tau : \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ to the *-homomorphism $\eta_\tau \colon \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes n}$ taking $a_1 \otimes \cdots \otimes a_m$ to the element $\hat{a}_1 \otimes \cdots \otimes \hat{a}_n$ with $\hat{a}_j = \begin{cases} a_i & \text{if } j = \tau(i) \\ 1_{\mathcal{A}} & \text{otherwise.} \end{cases}$

The colimit of $\mathcal{D}$ in $\mathbf{CSt}_{\mathrm{PU}}$ is $C(S(\mathcal{A}))$, with injections given by

$$\begin{aligned} \epsilon_n \colon \mathcal{A}^{\otimes n} &\to C(S(\mathcal{A})) \\ a &\mapsto \lambda\rho \in S(\mathcal{A}).\rho^{\otimes n}(a)\,. \end{aligned} \tag{4.32}$$

We instantiate this theorem with $\mathcal{A} = C(X)$. Note that either choice of tensor product could have been used since $C(X)$ is nuclear. Using proposition 3.45, the objects of the diagram become $C(X^n)$. The colimit object then is $CSC(X) \cong C(\mathcal{R}(X))$.

To find the explicit form of the morphisms of the colimit $C(X^n) \to C(\mathcal{R}(X))$, we chase clockwise around the following diagram:

$$\begin{tikzcd} C(X^n) \arrow[r, "\sim"] \arrow[d, dashed] & C(X)^{\otimes n} \arrow[d] \\ C(\mathcal{R}(X)) & C(S(C(X))). \arrow[l, "\sim"'] \end{tikzcd} \tag{4.33}$$

A functional $\phi\colon X^n \to \mathbb{C} \in C(X^n)$ becomes the unique element $\lim \sum_i \phi^i_1 \otimes \cdots \otimes \phi^i_n \in C(X)^{\otimes n}$ such that $\phi\,(x_1, \dots, x_n) = \lim \sum_i \prod_{j=1}^n \phi^i_j(x_j)$. The right-hand vertical arrow takes this to $\lambda\rho \in SC(X).\rho^{\otimes n}\,(\lim \sum_i \phi^i_1 \otimes \cdots \otimes \phi^i_n)$. Then the bottom isomorphism takes this to the functional on measures

$$\begin{aligned} \mu \mapsto & \int_{X^n} \lim \sum_i \phi^i_1 \otimes \cdots \otimes \phi^i_n \, \mathrm{d}\mu^n \\ &= \lim \sum_i \int_{X^n} \phi^i_1 \otimes \cdots \otimes \phi^i_n \, \mathrm{d}\mu^n \\ &= \lim \sum_i \prod_{j=1}^n \int_X \phi^i_j(x_j)\, \mathrm{d}\mu(x_j) \\ &= \int_{X^n} \lim \sum_i \prod_{j=1}^n \phi^i_j(x_j)\, \mathrm{d}\mu^n = \int_{X^n} \phi \, \mathrm{d}\mu^n . \end{aligned} \tag{4.34}$$

The limit can move in and out of the integral because all our functions are continuous on compact spaces and so convergence is dominated. Thus, the colimit morphisms $C(X^n) \to C(\mathcal{R}(X))$ take $\phi \in C(X^n)$ to the functional $\lambda\mu \in \mathcal{R}(X). \int_{X^n} \phi \, \mathrm{d}\mu^n$.

The morphisms of the diagram are more straightforward:

$$\begin{tikzcd}
C(X^m) \arrow[r, dashed] \arrow[d, "\wr"] & C(X^n) \\
C(X)^{\otimes m} \arrow[r, "\eta_\tau"'] & C(X)^{\otimes n} \arrow[u, "\wr"']
\end{tikzcd} \tag{4.35}$$

Following the diagram anti-clockwise from the top left takes

$$\begin{aligned}
\phi &\mapsto \lim \sum_i \phi_1^i \otimes \cdots \otimes \phi_m^i \\
&\mapsto \lim \sum_i \hat{\phi}_1^i \otimes \cdots \otimes \hat{\phi}_n^i \\
&\mapsto \lim \sum_i \prod_{j=1}^n \hat{\phi}_j^i(x_j)
\end{aligned} \tag{4.36}$$

and note that

$$\begin{aligned}
\lim \sum_i \prod_{j=1}^n \hat{\phi}_j^i(x_j) &= \lim \sum_i \prod_{j=1}^n \phi_j^i(x_{\tau(j)}) \\
&= \phi(x_{\tau(1)}, \ldots, x_{\tau(n)})
\end{aligned} \tag{4.37}$$

since $\hat{\phi}_j^i(x_j) = \begin{cases} \phi_k^i\left(x_{\tau(k)}\right) & \text{if } j = \tau(k) \\ 1 \quad \text{otherwise.} \end{cases}$

Consolidating, the following diagram is a colimit diagram in $\mathbf{CSt}_{\mathrm{PU}}$:

$$\begin{tikzcd}
C(X) \arrow[loop above] \arrow[r] \arrow[rrrrd, bend right] & C(X^2) \arrow[loop above] \arrow[r, shift left] \arrow[r, shift right] \arrow[rrrd, bend right] & C(X^3) \arrow[loop above] \arrow[loop above, out=60, in=120] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[rrd, bend right] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & C(X^n) \arrow[loop above] \arrow[loop above, out=60, in=120] \arrow[loop above, out=70, in=110] \arrow[loop above, out=80, in=100] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d] & \cdots \\
& & & & C(\mathcal{R}(X)) &
\end{tikzcd} \tag{4.38}$$

where the colimit maps $C(X^n) \to C(\mathcal{R}(X))$ are given by $\lambda\phi \in C(X^n), \mu \in \mathcal{R}(X). \int_{X^n} \phi \, \mathrm{d}\mu^n$, and for each injection $\tau \in \mathbf{Inj}(m, n)$ there is a map of the diagram $C(X^m) \to C(X^n)$ given by $- \circ X^\tau = \lambda\phi \in C(X^m). \phi \circ X^\tau$.

This colimit is preserved (i.e. becomes a limit in **ConvCH**) by the state-space functor $S \colon \mathbf{CSt}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{ConvCH}$. This will be the limit as stated in the theorem. Let us understand why.

The state-space functor acts on morphisms by pre-composition: $S(f) = - \circ f$. Applying S to the diagram, gives a diagram in **ConvCH** with objects $SC(X^n) \cong$

$\mathcal{R}(X^n)$. The limit becomes $SC(\mathcal{R}(X)) \cong \mathcal{R}^2(X)$. The morphisms of the diagram become morphisms $\mathcal{R}(X^n) \to \mathcal{R}(X^m)$. For an injection $\tau \in \mathbf{Inj}(m, n)$, the morphism of the diagram is given by following the square below:

$$\begin{tikzcd}
\mathcal{R}(X^m) & \mathcal{R}(X^n) \arrow[l, dashed] \arrow[d, "\wr"] \\
SC(X^m) \arrow[u, "\wr"'] & SC(X^n). \arrow[l]
\end{tikzcd} \tag{4.39}$$

Beginning with a measure $\mu \in \mathcal{R}(X^n)$, this becomes the state $\phi \in C(X^n) \mapsto \int_{X^n} \phi \, \mathrm{d}\mu$. The bottom morphism is pre-composition with the corresponding morphism in diag. (4.38), and as such a functional $\varphi \in C(X^n)$ is taken to $\int_{X^m} \varphi \circ X^\tau \, \mathrm{d}\mu = \int_{X^m} \varphi \, \mathrm{d}X^\tau_* \mu$.

The resultant measure in $\mathcal{R}(X^m)$ is $X^\tau_* \mu$, the pushforward of μ by X^τ.

The limit maps are found by following the diagram below anticlockwise, where the bottom morphism is pre-composition by the colimit morphisms in diag. (4.38) which take $\phi \in C(X^n)$ to the functional $\lambda \mu \in \mathcal{R}(X). \int_{X^n} \phi \, \mathrm{d}\mu^n$.

$$\begin{tikzcd}
\mathcal{R}^2(X) \arrow[r, dashed] \arrow[d, "\wr"] & \mathcal{R}(X^n) \\
SC(\mathcal{R}(X)) \arrow[r] & SC(X^n) \arrow[u, "\wr"']
\end{tikzcd} \tag{4.40}$$

Begin with Φ in the top left. In $SC(\mathcal{R}(X))$ we get the state $\int_{\mathcal{R}(X)} - \, \mathrm{d}\Phi$ on $C(\mathcal{R}(X))$.

The pre-composition takes this to $\phi \mapsto \int_{\mu \in \mathcal{R}(X)} \left[\int_{X^n} \phi \, \mathrm{d}\mu^n\right] \mathrm{d}\Phi$ for $\phi \in C(X^n)$. The corresponding measure in $\mathcal{R}(X^n)$ is $\int_{\mu \in \mathcal{R}(X)} \mu^n \left(-\right) \, \mathrm{d}\Phi$, since for $\psi = \sum_i a_i \chi_{A_i} \in C(X^n)$

$$\begin{aligned}
\int_{X^n} \psi \, \mathrm{d} \left[\int_{\mu \in \mathcal{R}(X)} \mu^n \left(-\right) \, \mathrm{d}\Phi\right] &= \sum_i a_i \int_{\mu \in \mathcal{R}(X)} \mu^n \left(A_i\right) \, \mathrm{d}\Phi \\
&= \int_{\mu \in \mathcal{R}(X)} \sum_i a_i \mu^n \left(A_i\right) \, \mathrm{d}\Phi \\
&= \int_{\mu \in \mathcal{R}(X)} \left[\int_{X^n} \psi \, \mathrm{d}\mu^n\right] \mathrm{d}\Phi
\end{aligned} \tag{4.41}$$

and so $\int_{X^n} \phi \, \mathrm{d} \left[\int_{\mu \in \mathcal{R}(X)} \mu^n \left(-\right) \, \mathrm{d}\Phi\right] = \int_{\mu \in \mathcal{R}(X)} \int_{X^n} \phi \, \mathrm{d}\mu^n \, \mathrm{d}\Phi$, for any $\phi \in C(X)$, by dominated convergence.

So the diagram is indeed the one in the statement of the theorem, and its limit is $\mathcal{R}^2(X)$, as expected. ☐

We may now prove theorem 4.3 in $\mathcal{K}l(\mathcal{R})$.

Proof of theorem 4.3. Noting then that $\mathrm{m}_{\mathcal{R}} \circ \mathcal{R}(\delta \circ X^{\tau}) = \mathcal{R}(X^{\tau})$ by the monad unit laws, and $\mathrm{m}_{\mathcal{R}} \circ \mathcal{R}(\mathrm{iid}_n)(\Phi) = \mathrm{m}_{\mathcal{R}}\left((\mathrm{iid}_n)_* \Phi\right) = \int_{\mathcal{R}(X)} \mu(-)\,\mathrm{d}\,(\mathrm{iid}_n)_* \Phi = \int_{\mathcal{R}(X)} \mu^n(-)\,\mathrm{d}\Phi$, the diagram of theorem 4.4 is exactly that of this theorem played in **ConvCH** under the inclusion functor $\mathcal{K}l(\mathcal{R}) \to$ **ConvCH** discussed in section 4.2.1.

The inclusion is full and faithful, as such the limit reflects. This concludes the proof of the categorical form of the Hewitt-Savage-de Finetti theorem. ☐

4.3 The Multiset de Finetti Limit

This section adapts theorem 4.3 with the use of multisets. This presents another way of encoding exchangeability categorically, now into the objects of the diagram. Recall from definition 2.20 that a multiset of elements of a set A is a collection of elements of A with, unlike a subset, multiple copies of each element permitted. Further, recall that a morphism $X \rightsquigarrow Y$ in the Kleisli category $\mathcal{K}l(\mathcal{R})$ is called deterministic if it is the lift of a morphism $f\colon X \to Y$ of the underlying topological spaces in **CH**.

In section 4.2, we followed three steps for describing exchangeable sequences of measures on a compact Hausdorff space X categorically to formulate a de Finetti construction as a limit in the Kleisli category of a probability monad.

1. It was first noted that a sequence of measures $\mu_n \in \mathcal{R}(X^n)$ is given by a (probabilistic) cone

$$\begin{tikzcd}
X & X^2 & X^3 & \cdots & X^n & \cdots \\
 & & & & \{*\} . \arrow[llllu, rightsquigarrow, "\mu_1"] \arrow[lllu, rightsquigarrow, "\mu_2"] \arrow[llu, rightsquigarrow, "\mu_3"] \arrow[u, rightsquigarrow, "\mu_n"]
\end{tikzcd} \tag{4.42}$$

2. In order that this sequence is *consistent*, the (deterministic) projection maps $X^{n+1} \to X^n$ are introduced.

$$\begin{tikzcd}
X & X^2 \arrow[l] & X^3 \arrow[l] & \cdots \arrow[l] & X^n \arrow[l] & \cdots \arrow[l] \\
&&&& \{*\} . \arrow[ullll, squiggly, bend left=15, "\mu_1"'] \arrow[ulll, squiggly, bend left=15, "\mu_2"'] \arrow[ull, squiggly, bend left=15, "\mu_3"'] \arrow[u, squiggly, "\mu_n"]
\end{tikzcd} \tag{4.43}$$

3. Finally the condition of *symmetry* or *permutation-invariance* is added in the form of the (again, deterministic) braiding maps $X^\sigma \colon X^n \to X^n$ for each $\sigma \in \mathcal{S}_n$ and their resulting compositions with projections.

$$\begin{tikzcd}
X \arrow[loop above] & X^2 \arrow[loop above] \arrow[loop above] \arrow[l, shift left] \arrow[l, shift right] & X^3 \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] & \cdots \arrow[l, shift left=3] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=3] & X^n \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[l, shift left=3] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=3] & \cdots \arrow[l, shift left=3] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=3] \\
&&&& \{*\} . \arrow[ullll, squiggly, bend left=15, "\mu_1"'] \arrow[ulll, squiggly, bend left=15, "\mu_2"'] \arrow[ull, squiggly, bend left=15, "\mu_3"'] \arrow[u, squiggly, "\mu_n"]
\end{tikzcd} \tag{4.44}$$

4. Replacing $\{*\}$ with a parametrizing space Y then yields a general cone and from here it is a matter of finding the universal such cone, the limit of the diagram.

The fact that the second step considering consistency precedes exchangeability in the third reflects the fact that the originally quantum de Finetti theorems of chapter 3 considered the space of exchangeable states on a C*-algebra $\mathcal{A}$, $I(\mathcal{A})$, as a subspace of the space of all states on its infinite tensor power, $S\left(\mathcal{A}^{\otimes\mathbb{N}}\right)$. In the classical case, this means considering the space of exchangeable measures on $X \in \mathbf{CH}$ as a subspace of the space of measures on its infinite product, $\mathcal{R}\left(X^{\mathbb{N}}\right)$. It is for this reason that we were concerned with the quantum Kolmogorov extension theorem (corollary 3.62) and its classical analogue (theorem 2.30), these being exactly the theorems necessary to construct such states and measures pointwise, and this guided the decision to prove categorical Kolmogorov extensions theorems in both settings, theorem 3.71 and theorem 4.2, before moving to de Finetti limits.

This angle is successful, but is by no means the only order of operations. As considered by Jacobs and Staton, multisets introduce a way of reordering the approach, swapping consistency and exchangeability and treating the latter as of primary concern. This follows in de Finetti's footsteps. Indeed, of all the de Finetti theorems in this work the one in this section seems most faithful to de Finetti's philosophy of probability: for de Finetti, the reason for considering exchangeability is not as a large scale assumption placed onto the outcome space of n-tuples but instead an internal acknowledgement of the subjective observers' inability to distinguish the likelihoods of different permutations of the same finite lists of results. In other words, the experimenter, judging the likelihood of events, is only capable of counting outcomes, and gains no information from their order. Supposing to begin with that we are working with a finite, discrete space $X \in \mathbf{CH}$, which is to say a set, so the sets $\mathcal{M}_n(X)$ are also finite and can be considered as discrete spaces, then the above sounds much less like a distribution on X^n which happens to be permutation invariant and much more like a distribution on $\mathcal{M}_n(X)$. Instead of steps 1 and 3 above then, we may approach a diagrammatic de Finetti theorem by taking as primitive a belief about the likelihood of different bags of results. Such a distribution, encoding symmetry in each $n \in \mathbb{N}$ would be given by a diagram of the form

$$\begin{tikzcd}
\mathcal{M}_1(X) & \mathcal{M}_2(X) & \mathcal{M}_3(X) & \cdots & \mathcal{M}_n(X) & \cdots \\
 & & & & \{*\}\,. \arrow[ullll, squiggly, bend left, "\mu_1"] \arrow[ulll, squiggly, bend left, "\mu_2"] \arrow[ull, squiggly, bend left, "\mu_3"] \arrow[u, squiggly, "\mu_n"]
\end{tikzcd} \tag{4.45}$$

How, then, to describe consistency? For the diagram of X^ns, consistency was a straightforward and deterministic evaluation. It represents a state of forgetting: flipping a coin n times is as good as flipping it $n+1$ times and forgetting the i^{th} result. Of course, this requires a choice of i, though any choice will do, and in the above, this choice was dealt with by including the projection $X^{n+1} \to X^n$ discarding the i^{th} result for every $i \in \{1, \ldots, n+1\}$. If our measures are no longer on X^n and X^{n+1}, but instead on $\mathcal{M}_n(X)$ and $\mathcal{M}_{n+1}(X)$, we can no longer take such an approach. There is no i^{th} result to drop, since order is no longer present. Instead, the diagram becomes probabilistic: draw an element at random, and drop it. This too has a de Finetti-like practical intuition to it: the statement of consistency originally was to say that one could choose to remove any particular experiment and maintain consistency, but the experiments themselves are assumed indistinguishable, and as such it makes sense to instead throw one out at random.

The corresponding map $\mathrm{DrawDel}_{n+1} \colon \mathcal{M}_{n+1}(X) \to \mathcal{M}_n(X)$ draws an element from a bag (exploiting the fact that we may turn a multiset into a distribution on its support by picking an element from it like an urn) and then produces a bag with that element removed. This is clearly probabilistic: given a multiset $\phi \in \mathcal{M}_{n+1}(X)$ and an element $x \in \mathrm{supp}(\phi)$, the probability that the result of drawing and deleting is the multiset

$$(\phi(x) - 1)\,|x\rangle + \sum_{\substack{a \in \mathrm{supp}(\phi) \\ a \neq x}} \phi(a)\,|a\rangle\,, \tag{4.46}$$

is $\frac{\phi(x)}{n+1}$, since this is the multiset resulting from dropping one $|x\rangle$ from ϕ.

In this setting, an exchangeable sequence of measures is given then by a diagram

$$\begin{tikzcd}
\mathcal{M}_1(X) & \mathcal{M}_2(X) \arrow[l, squiggly, "\mathrm{DrawDel}_2"'] & \mathcal{M}_3(X) \arrow[l, squiggly, "\mathrm{DrawDel}_3"'] & \ldots \arrow[l, squiggly, "\mathrm{DrawDel}_4"'] & \mathcal{M}_n(X) \arrow[l, squiggly, "\mathrm{DrawDel}_n"'] & \ldots \arrow[l, squiggly, "\mathrm{DrawDel}_{n+1}"'] \\
 & & & & \{*\}\,. \arrow[u, squiggly, "\mu_n"] \arrow[llu, squiggly, bend left, "\mu_3"] \arrow[lllu, squiggly, bend left, "\mu_2"] \arrow[llllu, squiggly, bend left, "\mu_1"] &
\end{tikzcd} \tag{4.47}$$

The subject of this section is cones of this form, replacing $\{*\}$ with a general parametrising space Y. Thus far, in this exposition we've really been working with the multiset monad on finite spaces X as sets, using finite distributions. In what follows, we extend the multiset monad to compact Hausdorff spaces, and follow the method above to generate a version of theorem 4.3, yet again realising $\mathcal{R}(X)$ as a de Finetti limit in $\mathcal{K}l(\mathcal{R})$, this time from a diagram of spaces of multisets.

4.3.1 Multisets on compact Hausdorff Spaces

Topological spaces admit quotients under group actions, as such, the construction of multisets as coequalisers extends naturally to **Top**:

Definition 4.5 (The Space of Multisets)**.** *Let X be a topological space and $n \in \mathbb{N}$. The* space of X-multisets of order n*,* $\mathcal{M}_n(X)$*, may be equivalently defined as*

1. *The quotient space X^n under the group action*

$$\begin{aligned} \mathcal{S}_n \times X^n &\to X^n \\ (\sigma, (x_1, \ldots, x_n)) &\mapsto \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right). \end{aligned} \tag{4.48}$$

2. *The coequaliser in* **Top** *of all the braiding maps $X^\sigma \colon X^n \to X^n$:*

$$\begin{tikzcd} X^n \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X^n \arrow[r, "\mathrm{acc}"] & \mathcal{M}_n(X). \end{tikzcd} \tag{4.49}$$

3. *The set $\mathcal{M}_n(X)$ as defined in definition 2.20 with the coarsest topology such that the accumulation map*

$$\begin{aligned} X^n &\to \mathcal{M}_n(X) \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^{n} |x_i\rangle \end{aligned} \tag{4.50}$$

   is continuous.

Proposition 4.6. *The above definitions are indeed equivalent.*

Proof. Fix $n \in \mathbb{N}$ and let us notate by $\mathcal{M}^1, \mathcal{M}^2$ and $\mathcal{M}^3$ the three definitions of $\mathcal{M}_n(X)$ in the order above.

Both $\mathcal{M}^1$ and $\mathcal{M}^2$ are quotients of X^n by equivalence relations (the latter as colimits in **Top** are simply colimits in **Set** with the quotient topology). In the first, two points are equivalent if the group action can map one onto the other. That is $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if and only if there exists a $\sigma \in \mathcal{S}_n$ with $y_i = x_{\sigma(i)}$ for all $i = 1, \ldots, n$. In the second, two points are equivalent if they are mapped to each other by some X^σ. Clearly, these are the same equivalence relation and so their quotient is the same.

Since the continuous accumulation map $X^n \to \mathcal{M}^3$ is invariant under precomposition with X^σ for all $\sigma \in \mathcal{S}_n$, there is a unique continuous map $\iota\colon \mathcal{M}^2 \to \mathcal{M}^3$ such that taken an n-tuple to its equivalence class in $\mathcal{M}^2$ and then applying ι is the same as applying the accumulation map to it. This map ι takes an equivalence class in $\mathcal{M}^2$ to the map $\phi\colon X \to \mathbb{N}$ which counts occurrences of an element $x \in X$ in any representative of the equivalence class.

Under the axiom of choice, this map is injective, if two n-tuples have the same number of occurrences for each element of $x \in X$ there must be a permutation that takes one to the other; and surjective, since for each finitely supported $\phi\colon X \to \mathbb{N}$ one can form an n-tuple that accumulates to it, and then look at the equivalence class of that tuple in $\mathcal{M}^2$.

It remains only to show that ι is open. The topology on $\mathcal{M}^3$ is generated by those sets whose inverse images under the accumulation map are open. Notate by $q\colon X^n \to \mathcal{M}^2$ the quotient map and let $U \subset \mathcal{M}^2$ be open. Then

$$\begin{aligned}\iota(U) &= \{\iota([x]) \mid [x] \in U\} \\ &= \{\mathrm{acc}(x) \mid [x] \in U\} \\ &= \Big\{\mathrm{acc}(x) \,\Big|\, x \in q^{-1}(U)\Big\} \\ &= \mathrm{acc}\Big(q^{-1}(U)\Big).\end{aligned} \tag{4.51}$$

Then

$$\begin{aligned}\mathrm{acc}^{-1}(\iota(U)) &= \Big\{x \in X^n \,\Big|\, \mathrm{acc}(x)\ inacc\Big(q^{-1}(U)\Big)\Big\} \\ &= \Big\{x \in X^n \,\Big|\, x \in q^{-1}(U)\Big\} \\ &= q^{-1}(U)\end{aligned} \tag{4.52}$$

which is open, giving $\iota(U)$ open as desired. Thus, ι is a homeomorphism and is the unique isomorphism that witnesses $\mathcal{M}^3$ as a coequaliser of the maps X^σ. □

Proposition 4.7. *For fixed $n \in \mathbb{N}$, the map $\mathcal{M}_n(X)$ with action on functions defined as in definition 2.23 is a functor.*

Proof. All that needs to be shown is the continuity of the map $\mathcal{M}_n(f)\colon \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ for a continuous function of topological spaces $X \to Y$.

Note that the composition $X^n \overset{f^n}{\to} Y^n \overset{\mathrm{acc}}{\to} \mathcal{M}_n(Y)$ is continuous and invariant under the group action on X^n. By the universal property of the quotient, this map factors through $X^n \overset{\mathrm{acc}}{\to} \mathcal{M}_n(X)$ as desired. $\square$

Proposition 4.8. *If X is compact and Hausdorff, for any $n \in \mathbb{N}$ the space of multisets $\mathcal{M}_n(X)$ is compact and Hausdorff.*

Proof. Compactness is a standard topological result: quotients of compact spaces are compact.

Suppose X is Hausdorff and take $\phi_1, \phi_2 \in \mathcal{M}_n(X)$ distinct. The sets $\mathrm{acc}^{-1}(\phi_1)$ and $\mathrm{acc}^{-1}(\phi_2)$ are disjoint, finite and closed in X^n. Since a compact Hausdorff space is normal there are disjoint open sets U_1 and U_2 such that $\mathrm{acc}^{-1}(\phi_i) \subset U_i$.

For $\sigma \in \mathcal{S}_n$, the map $X^\sigma \colon X^n \to X^n$ is a homeomorphism (as a bijective continuous map in **CH**), so we define the opens

$$U_i' = \bigcap_{\sigma \in \mathcal{S}_n} X^\sigma \left(U_i\right) \subset X^n$$

and, since acc is a quotient map, the images $V_i = \mathrm{acc}\left(U_i'\right) \subset \mathcal{M}_n(X)$ are open. Since $\mathrm{acc}^{-1}(\phi_i) = X^\sigma\left(\mathrm{acc}^{-1}(\phi_i)\right) \subset X^\sigma\left(U_i'\right)$ we have $\phi_i \in V_i$. Further, $U_1' \cap U_2' = \varnothing$ and both U_1' and U_2' are invariant under X^σ for all $\sigma \in \mathcal{S}_n$, so for all such σ, $x_i \in U_i'$, $\sigma(x_1) \neq x_2$, which is to say that for all $x_i \in U_i$, $\mathrm{acc}(x_1) \neq \mathrm{acc}(x_2)$ and thus $V_1 \cap V_2 = \mathrm{acc}\left(U_1'\right) \cap \mathrm{acc}\left(U_2'\right) = \varnothing$. We have found disjoint separating opens. $\square$

4.3.2 The Draw-and-Delete and Order Maps

Instead of the projections $X^{n+1} \to X^n$, throwing away an element of a bag from $\mathcal{M}_{n+1}(X)$ to get one in $\mathcal{M}_n(X)$ must be done probabilistically. Since $\mathcal{M}_n(-)$ is a functor on **CH**, an appropriate means of introducing probability is to use the Radon monad.

Definition 4.9 (Draw and Delete Maps)**.** *Fix $X \in$ **CH** and $n \in \mathbb{N}$. The* draw-and-delete map *is the $\mathcal{K}\ell(\mathcal{R})$ morphism* $\mathrm{DrawDel}_{n+1} \colon \mathcal{M}_{n+1}(X) \rightsquigarrow \mathcal{M}_n(X)$ *which, given a multiset $\phi \in \mathcal{M}_{n+1}(X)$, takes it to the multiset*

$$(\phi(x) - 1)\,|x\rangle + \sum_{\substack{a \in \mathrm{supp}(\phi) \\ a \neq x}} \phi(a)\,|a\rangle \tag{4.53}$$

with probability $\frac{\phi(a)}{n+1}$*, making a uniform choice of element of the multiset and removing it.*

For example, if $X = \mathbf{2}$, then

$$\mathcal{M}_3(X) = \{3\,|0\rangle\,, 2\,|0\rangle + |1\rangle\,, |0\rangle + 2\,|1\rangle\,, 3\,|1\rangle\} \tag{4.54}$$

and for $\phi = 2\,|0\rangle + |1\rangle$, $\mathrm{DrawDel}_3(\phi)$ is $2\,|0\rangle$ with probability $\frac{1}{3}$ and $|0\rangle + |1\rangle$ with probability $\frac{2}{3}$, since this is the likelihood of drawing and removing an instance of 0 for the bag represented by ϕ, containing two 0s and one 1.

Proposition 4.10. *The draw-and-delete maps* $\mathrm{DrawDel}_{n+1}$ *are continuous for all* $n \in \mathbb{N}$*.*

Proof. It is sufficient to show that the composition $\mathrm{DrawDel}_{n+1} \circ \mathrm{acc}\colon X^{n+1} \to \mathcal{R}(\mathcal{M}_n(X))$ is continuous.

Then $\mathrm{DrawDel}_{n+1} \circ \mathrm{acc}\,(x_1, \ldots, x_{n+1}) \in \mathcal{R}(\mathcal{M}_n(X))$ considered as a state on $C(\mathcal{M}_n(X))$ takes a continuous function $f\colon \mathcal{M}_n(X) \to \mathbb{C}$ to

$$\frac{1}{n}\sum_{i=1}^{n} f(\mathrm{acc}(\hat{x_i})) \in \mathbb{C}, \tag{4.55}$$

where $\hat{x_i}$ denotes dropping the i^{th} entry from $x_1, \ldots, x_n$.

The inverse image of a basis open set $\mathrm{ev}_f^{-1}(\Omega) = \{\phi \in SC(\mathcal{M}_n(X)) \,|\, \phi(f) \in \Omega\}$ with $f\colon \mathcal{M}_n(X) \to \mathbb{C}$ continuous and $\Omega \subset \mathbb{C}$ open, in X is

$$\left\{(x_1, \ldots, x_n) \,\middle|\, \frac{1}{n}\sum_{i=1}^{n} f(\mathrm{acc}(\hat{x_i})) \in \Omega\right\}. \tag{4.56}$$

Since f, acc and dropping the i^{th} coordinate are all continuous, this is just the inverse image of Ω under a sum of continuous functions, and so is open. □

The process of accumulating a tuple $(x_1, \ldots, x_n) \in X^n$ into a multiset in $\mathcal{M}_n(X)$ was simple and deterministic. The inverse operation, of ordering a multiset $\phi \in \mathcal{M}_n(X)$ into a tuple in X^n is not, since the choice must be arbitrary, but in $\mathcal{K}l(\mathcal{R})$ the arbitrariness of this choice can be alleviated by making it at random.

Definition 4.11 (Ordering a Multiset)**.** *For* $n \in \mathbb{N}$*, the morphism* $\operatorname{ord}\colon \mathcal{M}_n(X) \rightsquigarrow X^n$ *chooses an order of the elements of a multiset uniformly at random.*

Explicitly, for a multiset $\phi \in \mathcal{M}_n(X)$*, the set* $\operatorname{acc}^{-1}(\phi) \subset X^n$ *is finite. Given any element* $(x_1, \dots, x_n) \in \operatorname{acc}^{-1}(\phi)$*, then*

$$\operatorname{ord}(\phi) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \delta_{\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)}. \tag{4.57}$$

This map is certainly not deterministic. Given the multiset $\phi = 2\,|0\rangle + |1\rangle \in \mathcal{M}_3(\mathbf{2})$ as before, the distribution $\operatorname{ord}(\phi)$ returns either $(1,0,0)$, $(0,1,0)$ or $(0,0,1) \in X^3$, each with probability $\frac{1}{3}$.

Proposition 4.12. *The ordering maps* $\operatorname{ord}\colon \mathcal{M}_n(X) \to X^n$ *are continuous for each* $n \in \mathbb{N}$*.*

Proof. The symmetrisation map

$$\begin{aligned} \operatorname{symm}\colon X^n &\to \mathcal{R}(X^n) \\ (x_1, \dots, x_n) &\mapsto \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \delta_{\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)} \end{aligned} \tag{4.58}$$

is continuous as a convex combination of the morphisms $\delta \circ X^\sigma\colon X^n \to \mathcal{R}(X^n)$. Since it is group action-invariant, it factors continuously through $\mathcal{M}_n(X)$. □

Lemma 4.13. *If* $f\colon Y \rightsquigarrow X^n$ *is invariant under the group action on* X^n*, which is to say that* $X^\sigma \circ f = f$ *for all* $\sigma \in \mathcal{S}_n$*, then* $\operatorname{symm} \circ f = f$*.*

Proof. That f is invariant under the group action means that the distribution $f(- \,|\, y) \in \mathcal{R}(X^n)$ is symmetric for all $y \in Y$. Thus, recalling the notation of Markov kernels from eq. (2.26):

$$\begin{aligned} \operatorname{symm} \circ f\,(A \,|\, y) &= \int_{X^n} \operatorname{symm}(A \,|\, x)\, f(\mathrm{d}x \,|\, y) \\ &= \int_{X^n} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \delta_{X^\sigma(x)}(A)\, f(\mathrm{d}x \,|\, y) \\ &= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \int_{X^n} \delta_{X^\sigma(x)}(A)\, f(\mathrm{d}x \,|\, y) \\ &= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \int_{X^n} \delta_x(A)\, f(\mathrm{d}x \,|\, y) \\ &= \int_{X^n} \delta_x(A)\, f(\mathrm{d}x \,|\, y) = f(A \,|\, y) \end{aligned} \tag{4.59}$$

as desired. □

Lemma 4.14. *In $\mathcal{K}\ell(\mathcal{R})$,* acc *is a left inverse to* ord*.*

Note, this says symm is a split idempotent in $\mathcal{K}\ell(\mathcal{R})$.

$$\begin{tikzcd}
X^n \arrow[loop left, squiggly, "\text{symm}"] \arrow[r, bend left, "\text{acc}"] & \mathcal{M}_n(X) \arrow[l, squiggly, bend left, "\text{ord}"] \arrow[loop right, "\text{id}"]
\end{tikzcd} \tag{4.60}$$

Proof. The composition takes

$$\begin{aligned}
\text{acc} \circ \text{ord}(\phi) &= \int_{X^n} \text{acc}(x)\, \text{ord}(\mathrm{d}\phi) \\
&= \int_{X^n} \text{acc}(x)\, \mathrm{d}\left[\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \delta_{\left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)}\right] \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \text{acc}\Big(x_{\sigma(1)}, \dots, x_{\sigma(n)}\Big) \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \text{acc}(x_1, \dots, x_n) \\
&= \text{acc}(x_1, \dots, x_n) = \phi
\end{aligned} \tag{4.61}$$

for any $\phi \in \mathcal{M}_n(X)$ and $(x_1, \dots, x_n) \in \text{acc}^{-1}(\phi)$. ☐

Proposition 4.15. *The map* $\text{ord}\colon \mathcal{M}_n(X) \to X^n$ *is the equaliser of the braiding maps* $X^\sigma\colon X^n \to X^n$ *in* $\mathcal{K}\ell(\mathcal{R})$*.*

$$\begin{tikzcd}
\mathcal{M}_n(X) \arrow[r, squiggly, "\text{ord}"] & X^n \arrow[r, shift left=2, "X^\sigma"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X^n.
\end{tikzcd} \tag{4.62}$$

Proof. Suppose we have another probabilistic process $f\colon Y \rightsquigarrow X^n$ such that $X^\sigma \circ f = f$ for all $\sigma \in \mathcal{S}_n$. Then $\text{acc} \circ f\colon Y \rightsquigarrow \mathcal{M}_n(X)$ is a morphism of cones by lemma 4.13:

$$\begin{tikzcd}
\mathcal{M}_n(X) \arrow[r, squiggly, "\text{ord}"] & X^n \arrow[r, shift left=2, "X^\sigma"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X^n \\
Y \arrow[u, squiggly, "\text{acc} \circ f"] \arrow[ur, squiggly, "f"'] & &
\end{tikzcd} \tag{4.63}$$

Further this map is the unique such morphism since, if $g\colon Y \rightsquigarrow \mathcal{M}_n(X)$ has $\text{ord} \circ g = f$, then composing with acc and applying lemma 4.14 gives $g = \text{acc} \circ f$. ☐

4.3.3 Hewitt-Savage-de Finetti as a Categorical Limit with Multisets

Theorem 4.16 (Categorical Hewitt-Savage-de Finetti Theorem with Multisets)**.** *Let X be a compact Hausdorff space. Consider the draw-and-delete diagram $(\mathbb{N}, \leq)^{\mathrm{op}} \to \mathcal{K}\ell(\mathcal{R})$ which takes n to $\mathcal{M}_n(X)$ and, for a natural number $n \in \mathbb{N}$, takes $n+1 \to n$ to the draw-and-delete map $\mathrm{DrawDel}_{n+1} \colon \mathcal{M}_{n+1}(X) \rightsquigarrow \mathcal{M}_n(X)$.*

The limit of this diagram is the space of measures on X, $\mathcal{R}(X)$, with limit projections $\mathrm{iid}_n^{\mathcal{M}} \colon \mathcal{R}(X) \rightsquigarrow \mathcal{M}_n(X)$ given by the composition

$$\mathcal{R}(X) \overset{\mathrm{iid}_n}{\rightsquigarrow} X^n \xrightarrow{\mathrm{acc}} \mathcal{M}_n(X). \tag{4.64}$$

Proof. Note that $m \leq n$ if and only if $\mathbf{Inj}(m, n) \neq \varnothing$, and so there is a functor $\mathbf{Inj} \to (\mathbb{N}, \leq)$. As such, we may consider the diagram above as being $\mathbf{Inj}^{\mathrm{op}}$ shaped.

We show that the maps $\mathrm{ord} \colon \mathcal{M}_n(X) \rightsquigarrow X^n$ form a natural transformation of $\mathbf{Inj}^{\mathrm{op}}$-shaped diagram, where the latter is the diagram of the categorical Hewitt-Savage-de Finetti theorem (theorem 4.3). Clearly for a permutation $\sigma \in \mathcal{S}_n$, $X^\sigma \circ \mathrm{ord} = \mathrm{ord}$. For the projection $\pi_{n+1} \colon X^{n+1} \to X^n$, we wish to show that the following diagram commutes in $\mathcal{K}\ell(\mathcal{R})$:

$$\begin{tikzcd}
X^n & X^{n+1} \arrow[l, "\pi_{n+1}"'] \\
\mathcal{M}_n(X) \arrow[u, squiggly, "\mathrm{ord}"] & \mathcal{M}_{n+1}(X). \arrow[u, squiggly, "\mathrm{ord}"'] \arrow[l, squiggly, "\mathrm{DrawDel}_{n+1}"]
\end{tikzcd} \tag{4.65}$$

Take a multiset $\phi \in \mathcal{M}_{n+1}(X)$ and an element $(x_1, \ldots, x_n) \in X^n$ such that $\mathrm{acc}((x_1, \ldots, x_n)) = \phi - |a\rangle$ for some $a \in \mathrm{supp}(\phi)$. The anti-clockwise direction of the square orders ϕ at random into an $(n+1)$-tuple and then removes the last element. The probability that the result is $(x_1, \ldots, x_n)$ is $\frac{S_{(x_1,\ldots,x_n,a)}}{(n+1)!}$, where $S_{(x_1,\ldots,x_m)}$ is the number of permutations in $\mathcal{S}_m$ which fix $(x_1, \ldots, x_m)$.

The clockwise direction, on the other hand, removes an element at random from the multiset and then further gives it a random order. The probability that the result is $(x_1, \ldots, x_n)$ is $\frac{\phi(a)}{n+1} \times \frac{S_{(x_1,\ldots,x_n)}}{n!}$, and $S_{(x_1,\ldots,x_n,a)} = S_{(x_1,\ldots,x_n)}\phi(a)$, so the naturality diagram commutes.

$$\begin{tikzcd}
X \arrow[loop above] & X^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop above] \arrow[loop above] & X^3 \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \vdots \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots & \vdots \arrow[l, shift left=2] \arrow[l, shift right=2] & X^n \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] \arrow[loop above] & \vdots \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \\
X \arrow[u, squiggly, "\mathrm{ord}"] & \mathcal{M}_2(X) \arrow[l, squiggly, "\mathrm{DrawDel}_2"'] \arrow[u, squiggly, "\mathrm{ord}"] & \mathcal{M}_3(X) \arrow[l, squiggly, "\mathrm{DrawDel}_3"'] \arrow[u, squiggly, "\mathrm{ord}"] & & \cdots \arrow[ll] & & \mathcal{M}_n(X) \arrow[ll, squiggly] \arrow[u, squiggly, "\mathrm{ord}"] & & {} \arrow[ll, squiggly]
\end{tikzcd} \tag{4.66}$$

Via composition with this natural transformation, any cone of the draw-and-delete diagram also gives a cone of the X^n-exchangeability diagram. In particular, the universal cone of this form is the composition $\mathrm{ord} \circ \mathrm{iid}_n^{\mathcal{M}} = \mathrm{ord} \circ \mathrm{acc} \circ \mathrm{iid}_n = \mathrm{iid}_n \colon \mathcal{R}(X) \rightsquigarrow X^n$.

On the other hand, suppose $f_n \colon Y \rightsquigarrow X^n$ is a cone of the exchangeability diagram. Then, in particular, $X^\sigma \circ f_n = f_n$ for all $\sigma \in \mathcal{S}_n$, and by the universal property of the coequaliser there exists a unique map $F_n \colon Y \rightsquigarrow \mathcal{M}_n(X)$ such that $f_n = \mathrm{ord} \circ F_n$ for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
X^n & & X^{n+1} \arrow[ll, "\pi_{n+1}"'] \\
& Y \arrow[ul, squiggly, "f_n"] \arrow[ur, squiggly, "f_{n+1}"'] \arrow[dl, dashed, "F_n"] \arrow[dr, dashed, "F_{n+1}"'] & \\
\mathcal{M}_n(X) \arrow[uu, squiggly, "\mathrm{ord}"] & & \mathcal{M}_{n+1}(X). \arrow[ll, squiggly, "\mathrm{DrawDel}_{n+1}"] \arrow[uu, squiggly, "\mathrm{ord}"']
\end{tikzcd} \tag{4.67}$$

Via the naturality of ord, the outer square commutes, and thus so do all the triangles, including the lower one, which is to say that $\{F_n\}$ is a cone of the draw-and-delete diagram. Thus, both diagrams have the same limit. □

4.3.4 Relationship to Jacobs and Staton's de Finetti Limit

Though theorems 4.3, 4.4 and 4.16 are couched in the same language as Staton and Jacobs in [91], they exist in a different category. It is worth asking, in what way are these results related?

Note that in Jacobs and Staton, a theorem of the form of theorem 4.16 was proved. Their main theorem was proved for $X = \mathbf{2} \in \mathbf{Meas}$, in the Kleisli category of the Giry monad, of which $\mathcal{K}l(\mathcal{R})$ is a subcategory. Since $\mathbf{2} := \{0,1\}$ is a discrete space, both in **Meas** and in **CH**, the diagram of which the limit is being taken, of the finite spaces $\mathcal{M}_n(\mathbf{2})$ lives in $\mathcal{K}l(\mathcal{R}) \subset \mathcal{K}l(\mathcal{G})$. Further, the spaces $\mathcal{G}(\mathbf{2})$ and $\mathcal{R}(\mathbf{2})$ are both isomorphic to $[0,1]$ as measurable spaces. Thus, the limit cone looks the same in both categories. The divergence occurs though in the fact that Jacobs and Staton allow for *measurable* parametrised exchangeable sequences $\mathcal{X} \to \mathcal{G}(\mathbf{2}^n)$, with resultant measurable mediating maps $\mathcal{X} \to \mathcal{G}^2(\mathbf{2})$, whereas the parametrisations in theorem 4.16 must obey the stronger condition of being *continuous* maps $Y \to \mathcal{R}(\mathbf{2}^n)$. Not only is this class of permissible maps different even when $\mathcal{X} = Y$, but $\mathcal{X}$ may be a general measurable space that does not arise as a topological space with the Borel σ-algebra.

In particular, they use Pólya's urn as an example throughout their paper. This example is not stateable in our setting: the parameters are the initial number of black and white balls, and can take values in $\mathbb{N} \times \mathbb{N}$, which is non-compact when considered as a discrete space. There may be workarounds: we *can*, for example, consider Polya's urn as $\{1,\ldots,N\}^2 \rightsquigarrow \mathbf{2}^n$, where $N \in \mathbb{N}$ is just sufficiently large as to describe any combination of white and black balls we might use, or perhaps consider compactification procedures on $\mathbb{N}^2$, but the general complaint remains.

On the other hand, and the main contribution in this area, is that our result extends beyond $X = \mathbf{2}$ to limits for **all** compact Hausdorff spaces $X \in \mathbf{CH}$, and as such provides a universal property for the space of Radon measures on X, $\mathcal{R}(X)$, for any such X. The methods used towards the theorems of this work, namely relationships to C*-algebras and the chain of monadic functors $\mathcal{E}m(\mathcal{R}) \to \mathbf{CH} \to \mathbf{Set}$, are not transferable to **Meas**, the domain of $\mathcal{G}$. Similarly, the Kolmogorov extension theorem and the de Finetti theorems often have topological constraints, so limit theorem for general $\mathcal{X} \in \mathbf{Meas}$ would have to take a different approach. An alternative avenue of exploration would be whether the existence of such Kolmogorov and de Finetti limits might be a universal property of the Radon monad, distinguishing it from other probability monads, though there is not enough information to know yet if this is reasonable.

4.4 The Symmetric Tensor Quantum de Finetti Colimit

Given that theorems 4.3 and 4.4 are commutative instantiations of the categorical quantum de Finetti theorem 3.73, it is reasonable to ask whether theorem 4.16 is also a version of a quantum result. In this section, it is illustrated that it is.

All the definitions and theorems here are dual to definitions and results in section 4.3 when $\mathcal{A}$ is a commutative C*-algebra. Such dualities are pointed out in titles.

In what follows, let $\mathcal{A}^{\otimes\mathbb{N}} \in \left\{\mathcal{A}^{\otimes\mathbb{N}}_{\text{min}}, \mathcal{A}^{\otimes\mathbb{N}}_{\text{max}}\right\}$ be a fixed choice of tensor product power.

Definition 4.17 (Algebra of Symmetric Tensors, dual to definition 4.5)**.** *Given a C*-algebra $\mathcal{A}$,* the algebra of symmetric tensors of $\mathcal{A}$ of order n *is the equaliser in* $\mathbf{CSt}_{\text{MIU}}$ *of all the braiding maps* $\mathcal{A}^{\otimes\sigma}\colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$*.*

$$\begin{tikzcd} \mathcal{Q}_n(\mathcal{A}) \arrow[r] & \mathcal{A}^{\otimes n} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & \mathcal{A}^{\otimes n} \end{tikzcd} \tag{4.68}$$

This somewhat obscures the nature (and perhaps even the existence) of the space, but working with C*-algebras instead of topological spaces makes it straightforward to calculate that $\mathcal{Q}_n(\mathcal{A})$ is the set of elements fixed by the action of $\mathcal{S}_n$: $\mathcal{Q}_n(\mathcal{A}) = \{a \in \mathcal{A}^{\otimes n} \mid \mathcal{A}^{\otimes\sigma}(a) = a, \forall\sigma \in \mathcal{S}_n\}$. The equaliser map is just inclusion.

Indeed, this subset is closed under the *-algebra operations, and the $\mathcal{A}^{\otimes\sigma}$ maps are continuous, so this set is also topologically closed too. As such, it forms a sub-C*-algebra of $\mathcal{A}^{\otimes n}$. If $\phi\colon \mathcal{B} \to \mathcal{A}^{\otimes n}$ is a *-homomorphism with $\mathcal{A}^{\otimes\sigma} \circ \phi = \phi$ for all $\sigma \in \mathcal{S}_n$, then we must have $\mathcal{A}^{\otimes\sigma}(\phi(b)) = \phi(b)$ for all $b \in \mathcal{B}$ and thus the range of ϕ lives in $\mathcal{Q}_n(\mathcal{A})$.

Definition 4.18 (Tensor Symmetrisation, dual to eq. (4.58))**.** *Let $\mathcal{A}$ be a C*-algebra. For $n \in \mathbb{N}$,* the $(n$-$)$symmetrisation map on $\mathcal{A}^{\otimes n}$ *is the positive, unital map*

$$\begin{aligned} \mathfrak{S}\colon \mathcal{A}^{\otimes n} &\to \mathcal{A}^{\otimes n} \\ a &\mapsto \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \mathcal{A}^{\sigma}(a). \end{aligned} \tag{4.69}$$

In particular, $\mathfrak{S}(a_1 \otimes \cdots \otimes a_n) = \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$.

Note that $\mathfrak{S}$ is clearly unital, and it is a convex combination of *-homomorphisms and thus is completely positive. It is also true that the restriction of $\mathfrak{S}$ to $\mathcal{Q}_n(\mathcal{A})$ is the identity and that $\operatorname{im}(\mathfrak{S}) = \mathcal{Q}_n(\mathcal{A})$, so we may consider $\mathfrak{S} \colon \mathcal{A}^{\otimes n} \to \mathcal{Q}_n(\mathcal{A})$. This is the dual statement to lemma 4.14.

Lemma 4.19 (Dual to lemma 4.13)**.** *If $\phi \colon \mathcal{A}^{\otimes n} \to \mathcal{B} \in \mathbf{CSt}_{\mathrm{PU}}$ is such that $\phi \circ \mathcal{A}^{\otimes \sigma} = \phi$ for all $\sigma \in \mathcal{S}_n$, then $\phi = \phi \circ \mathfrak{S}$.*

Proof.

$$\phi \circ \mathfrak{S} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \phi \circ \mathcal{A}^{\sigma} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \phi = \phi. \tag{4.70}$$

□

Proposition 4.20 (Algebra of Symmetric Tensors as a Coequaliser, dual to proposition 4.15)**.** *The map $\mathfrak{S} \colon \mathcal{A}^{\otimes n} \to \mathcal{Q}_n(\mathcal{A})$ is the coequaliser of the braiding maps $\mathcal{A}^{\otimes \sigma} \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$ in $\mathbf{CSt}_{\mathrm{PU}}$ and in $\mathbf{CSt}_{\mathrm{CPU}}$.*

$$\begin{tikzcd} \mathcal{A}^{\otimes n} \arrow[r, shift left=1.5] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.5] & \mathcal{A}^{\otimes n} \arrow[r, "\mathfrak{S}"] & \mathcal{Q}_n(\mathcal{A}). \end{tikzcd} \tag{4.71}$$

Proof. Suppose we have another positive, unital map $\phi \colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ such that $\phi \circ \mathcal{A}^{\otimes \sigma} = \phi$ for all $\sigma \in \mathcal{S}_n$. Then by lemma 4.19, $\phi \circ \mathfrak{S} = \phi$, which means the following is a morphism of cocones:

$$\begin{tikzcd} \mathcal{A}^{\otimes n} \arrow[r, shift left=1.5] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.5] & \mathcal{A}^{\otimes n} \arrow[r, "\mathfrak{S}"] \arrow[dr, "\phi"'] & \mathcal{Q}_n(\mathcal{A}) \arrow[d, "\phi\restriction_{\mathcal{Q}_n(\mathcal{A})}"] \\ & & \mathcal{B} \end{tikzcd} \tag{4.72}$$

Further, this is the unique such morphism: if $\psi \colon \mathcal{Q}_n(\mathcal{A}) \to \mathcal{B}$ has $\psi \circ \mathfrak{S} = \phi$ then for any $a \in \mathcal{Q}_n(\mathcal{A})$, $\psi(a) = \psi\left(\mathfrak{S}(a)\right) = \phi(a)$. □

To state the categorical quantum de Finetti theorem using these algebras an equivalent to the draw-and-delete maps between multisets are necessary. The intuition for these maps comes from the fact that the draw-and-delete maps may be defined using the randomised delete map $X^{n+1} \to X^n$ which uniformly chooses and entry of the $n+1$-tuple and removes it:

$$\begin{tikzcd}
\mathcal{M}_{n+1}(X) \arrow[r, rightsquigarrow, "\mathrm{DrawDel}_{n+1}"] & \mathcal{M}_n(X) \\
X^{n+1} \arrow[u, "\mathrm{acc}"] \arrow[r, rightsquigarrow, "\mathrm{RandDel}"'] & X^n. \arrow[u, "\mathrm{acc}"']
\end{tikzcd} \tag{4.73}$$

The dual diagram in $\mathbf{CSt}_{\mathrm{PU}}$ replaces the accumulation maps with the inclusion $\mathcal{Q}_{\mathcal{A}}(n) \hookrightarrow \mathcal{A}^{\otimes n}$. The bottom map (and thus also the top map) is then given by placing $1_{\mathcal{A}}$ into a tensor symmetrically. For example,

$$a_1 \otimes a_2 \mapsto \frac{1}{3}\left(1_{\mathcal{A}} \otimes a_1 \otimes a_2 + a_1 \otimes 1_{\mathcal{A}} \otimes a_2 + a_1 \otimes a_2 \otimes 1_{\mathcal{A}}\right) \tag{4.74}$$

Explicitly, let $\jmath_{n+1}^k \colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes(n+1)}$ be the morphism which inserts $1_{\mathcal{A}}$ at the k^{th} position in a tensor of the algebraic tensor product. Then the map we're interested in is defined as follows:

Definition 4.21 (Dual to definition 4.9). *For a C^*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, we define the completely positive, unital map*

$$J_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} \jmath_{n+1}^k \colon \mathcal{Q}_n(\mathcal{A}) \to \mathcal{Q}_{n+1}(\mathcal{A}). \tag{4.75}$$

Strictly, J_{n+1} is a map $\mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes(n+1)}$, but it is clear that it takes symmetric elements to symmetric elements. We are only interested in the restricted version.

Theorem 4.22 (Categorical Quantum de Finetti Theorem with Symmetric Tensors, dual to theorem 4.16). *Let $\mathcal{A}$ be a C^*-algebra. Consider the diagram $(\mathbb{N}, \leq) \to \mathbf{CSt}_{\mathrm{PU}}$ which takes n to $\mathcal{Q}_n(\mathcal{A})$ and, for a natural number $n \in \mathbb{N}$, takes $n \to n+1$ to $J_{n+1} \colon \mathcal{Q}_n(\mathcal{A}) \to \mathcal{Q}_{n+1}(\mathcal{A})$.*

The colimit of this diagram is the C^*-algebra $C(S(\mathcal{A}))$, with colimit injections given by

$$\begin{aligned} \epsilon_n \colon \mathcal{Q}_n(\mathcal{A}) &\to C(S(\mathcal{A})) \\ a &\mapsto \left(\rho \mapsto \rho^{\otimes n}(a)\right). \end{aligned} \tag{4.76}$$

The same result is true in $\mathbf{CSt}_{\mathrm{CPU}}$.

Proof. The proof is identical to that of theorem 4.16. We again consider the diagram above as being **Inj**-shaped. Then the maps $\mathfrak{S}\colon \mathcal{A}^{\otimes n} \to \mathcal{Q}_n(\mathcal{A})$ form a natural transformation of **Inj**-shaped diagrams, where the latter is the diagram of the categorical quantum de Finetti theorem in $\mathbf{CSt}_{\mathrm{PU}}$. For a permutation $\sigma \in \mathcal{S}_n$, clearly $\mathfrak{S} \circ \mathcal{A}^{\otimes\sigma} = \mathfrak{S}$. For each $\jmath^k_{n+1}\colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes(n+1)}$ the diagram

$$\begin{tikzcd}
\mathcal{A}^{\otimes n} \arrow[r, "\jmath^k_{n+1}"] \arrow[d, "\mathfrak{S}"'] & \mathcal{A}^{\otimes(n+1)} \arrow[d, "\mathfrak{S}"] \\
\mathcal{Q}_n(\mathcal{A}) \arrow[r, "J_n"'] & \mathcal{Q}_{n+1}(\mathcal{A})
\end{tikzcd} \tag{4.77}$$

commutes via straightforward computation. The clockwise direction adds $1_\mathcal{A}$ into a tensor and then symmetrises its elements. The anticlockwise direction symmetrises a tensor, and then adds $1_\mathcal{A}$ at every place in every vector of the linear combination.

$$\begin{tikzcd}
\mathcal{A} \arrow[loop above] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mathfrak{S}"] & \mathcal{A}^{\otimes 2} \arrow[loop, out=135, in=90] \arrow[loop, out=90, in=45] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] \arrow[d, "\mathfrak{S}"] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{A}^{\otimes n} \arrow[loop, out=150, in=120] \arrow[loop, out=120, in=90] \arrow[loop, out=90, in=60] \arrow[loop, out=60, in=30] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\mathfrak{S}"] & \cdots \\
\mathcal{A} \arrow[r, "J_1"'] & \mathcal{Q}_2(\mathcal{A}) \arrow[r, "J_2"'] & \cdots \arrow[r, "J_{n-1}"'] & \mathcal{Q}_n(\mathcal{A}) \arrow[r, "J_n"'] & \cdots .
\end{tikzcd} \tag{4.78}$$

Via composition with this natural transformation, any cocone of the symmetric tensor diagram also gives a cocone of the $\mathcal{A}^{\otimes n}$-exchangeability diagram. In particular, the universal cone of this form is the composition $\epsilon_n \circ \mathfrak{S}$.

On the other hand, suppose $\phi_n\colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ is a cocone of the exchangeability diagram. Then, in particular, $\phi_n \circ \mathcal{A}^{\otimes\sigma} = \phi_n$ for all $\sigma \in \mathcal{S}_n$, and by the universal property of the equaliser there exists a unique map $\Phi_n\colon \mathcal{Q}_n(\mathcal{A}) \to \mathcal{B}$ such that $\phi_n = \Phi_n \circ \mathfrak{S}$ for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
\mathcal{A}^{\otimes n} \arrow[rr, "\jmath^k_{n+1}"] \arrow[dr, "\phi_n"] \arrow[dd, "\mathfrak{S}"'] & & \mathcal{A}^{\otimes(n+1)} \arrow[dl, "\phi_{n+1}"] \arrow[dd, "\mathfrak{S}"] \\
& \mathcal{B} & \\
\mathcal{Q}_n(\mathcal{A}) \arrow[rr, "J_n"'] \arrow[ur, dashed, "\Phi_n"] & & \mathcal{Q}_{n+1}(\mathcal{A}). \arrow[ul, dashed, "\Phi_{n+1}"']
\end{tikzcd} \tag{4.79}$$

Via the naturality of $\mathfrak{S}$, the outer square commutes, and thus so do all the triangles, including the lower one, which is to say that $\{\Phi_n\}$ is a cocone of the symmetric tensor diagram. Thus, both diagrams have the same colimit.

For the statement in $\mathbf{CSt}_{\mathrm{CPU}}$, it is sufficient to notice that the colimit maps are completely positive, as are the maps $\mathfrak{S}$ and J_n. $\square$

Instantiating this result with a commutative C*-algebra $C(X)$ gives an alternative proof of theorem 4.16.

Fock Space These spaces of symmetric tensors are relevant in the study of quantum states of indistinguishable particles. In such a context, a *particle* is a Hilbert space of a single quantum system $\mathcal{H}$. A state of two such particles would be in $S(\mathcal{B}(\mathcal{H} \otimes_H \mathcal{H}))$, but if these two particles are indistinguishable then we should only permit states which are symmetric when the two copies of $\mathcal{H}$ in $\mathcal{B}(\mathcal{H} \otimes_H \mathcal{H})$ are exchanged. This would be $S(\mathcal{Q}_2(\mathcal{B}(\mathcal{H})))$.

The (symmetric) Fock space of $\mathcal{H}$ is the Hilbert space completion of the direct sum of all these $\mathcal{Q}_n(\mathcal{B}(\mathcal{H}))$ spaces [See, for example, section 10.6 of Landsman 114].

In this work, the use of the Fock space has not been explored. This is largely because the original quantum de Finetti theorems did not use this lens, but it might be an interesting site for future work.

Additionally, the use of Fock spaces as models of the linear logic bang modality "!", and more generally the use of free symmetric algebras for this same goal [15], suggests that perhaps the results above might be of use in that context.

This concludes chapter 4. Using the quantum de Finetti theorem of chapter 3, we have proved two categorical de Finetti theorems in the style of Hewitt and Savage, classifying $\mathcal{R}(X)$, the space of measures on a compact Hausdorff space X as the limit of two diagrams, one of which used exchangeable sequence of measures on X^n, the other used measures on spaces of multisets of elements of X, $\mathcal{M}_n(X)$. This result inspired section 4.4, where a new categorical quantum de Finetti theorem, using a diagram of spaces of symmetric tensors, classified $CS(\mathcal{A})$ as a colimit.

5

De Finetti Theorems As Final Coalgebra and Initial Transition Algebra

Contents

5.1 Introduction

The content of this chapter enables another classification of the space of measures $\mathcal{R}(X)$ on a compact Hausdorff space X, this time as a final coalgebra of the functor $X \times -$ in the Kleisli category of $\mathcal{R}$. The approach to this is very categorical: most of the results of this chapter are proved in section 5.2 for a general affine, commutative monad $\mathcal{T}$ on a Cartesian category $\mathbf{C}$. In section 5.3, these are specialised to the case of the Radon monad $\mathcal{R}$ on $\mathbf{CH}$; the reasoning in this chapter would also be valid for any monad with similar limit forms of the Kolmogorov extension theorem and de Finetti theorem as shown those shown in chapter 4 with a simple monomorphism-preservation condition. Note that the categorical coalgebras of interest are unrelated to the coalgebras of linear algebra.

The main classical results in this chapter are a co-algebraic Kolmogorov extension theorem, theorem 5.7, showing that if a Kolmogorov extension theorem-like limit $A^{\mathbb{N}}$ exists in $\mathbf{C}$ and $\mathcal{K}l(\mathcal{T})$, then the functor $A \boxtimes -$ in $\mathcal{K}l(\mathcal{T})$, using the monoidal structure inherited from $\mathbf{C}$, has a final coalgebra, *the stream coalgebra* (definition 5.6), carried by $A^{\mathbb{N}}$; and theorem 5.19, which states that in $\mathcal{K}l(\mathcal{R})$, or more generally, whenever both the Kolmogorov extension theorem limit and de Finetti limit exist in $\mathcal{K}l(\mathcal{T})$ and the above-mentioned monomorphism-preservation condition holds, there is a final exchangeable coalgebra of the functor $X \times -$, called *the sample coalgebra* (definition 5.15) carried by $\mathcal{R}(X)$, and defined using the strength of the monad. This result in full generality is laid out in theorem 5.21, to demonstrate that similar de Finetti limit theorems, like that of Jacobs and Staton in $\mathcal{K}l(\mathcal{G})$ also give rise to these final coalgebras.

This approach also inspires a non-commutative version in section 5.5. The main result of this section is theorem 5.32: a quantum version of the coalgebra de Finetti theorem, this time classifying the space of continuous functions on states of a C*-algebra $CS(\mathcal{A})$ as the carrier of an initial transition algebra structure for the functor $\mathcal{A} \mathbin{\hat{\otimes}} -$ in $\mathbf{CSt}_{\mathrm{CPU}}$ under either monoidal structure.

This is particularly interesting because $\mathbf{CSt}_{\mathrm{CPU}}$ has none of the categorical machinery that is used to proof the commutative version in $\mathcal{K}l(\mathcal{R})$. Instead, the initial transition algebra must be defined directly, though the angle of attack for the proof is inspired by the commutative case. There is a quantum transition algebra Kolmogorov extension theorem, theorem 5.24, showing that $\mathcal{A}^{\otimes\mathbb{N}}$ is the carrier of the structure of initial transition algebra, called the *Kolmogorov transition algebra* (definition 5.23) whilst the initial exchangeable transition algebra, *the evaluation transition algebra* (definition 5.25) has the form $\gamma^{\text{ev}}\colon \mathcal{A} \mathbin{\hat{\otimes}} CS(\mathcal{A}) \to CS(\mathcal{A})$.

5.1.1 Motivating Coalgebras

If $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, a *coalgebra of F* or an *F-coalgebra* is a morphism $A \to FA$ in $\mathbf{D}$.

Coalgebras of this form are used extensively in computer science for studying transition systems, and have a rich theory [See, e.g., 66, 90, 134]. Morphisms of coalgebras are used to understand relationships between transition processes, and final coalgebras are understood as a sort of universe of all possible behaviours [2]. The unique morphism from a coalgebra to the final coalgebra picks out its particular behaviour, and using this a notion of behavioural equivalence can be developed based on whether two coalgebras have the same image in the final coalgebra.

These methods have frequently been used in the study of probabilistic systems [for a brief survey, see 142, 143, 155]. Coalgebras in Kleisli categories of monads are used to model effectful computation, of which probabilistic programming is one example, and as such coalgebras in the Kleisli category of probability monads are often the objects of study [67]. This chapter continues in this tradition, approaching exchangeability from the perspective of coalgebra.

We are interested in coalgebras of the functor $X \times -$ in $\mathcal{K}l(\mathcal{R})$, the Kleisli category of the Radon monad. Such a coalgebra is morphism $c\colon Y \rightsquigarrow X \times Y$, which uses an element $y \in Y$ that may be understood as a parameter for some kind of experiment or a state of a system, to generate probabilistically a result $x \in X$ and an updated state on Y.

Of course, given such a c, repeated application of c to the returned value in Y creates a stream of results in X using an initial seed value from Y and as such gives maps $c^{(n)}\colon Y \rightsquigarrow X^n \times Y$ for the n^{th} such iterant.

For example, taking $Y = [0,1]$ and $X = \mathbf{2} := \{0,1\}$, a coalgebra $c\colon [0,1] \rightsquigarrow \mathbf{2} \times [0,1]$ can describe a simple slot machine which, when beginning with a certain chance of winning $p \in [0,1]$, flips a coin of bias p to see if the user wins, and updates the probability of winning such that a user is unlikely to stay on a losing streak, making a win more likely after a loss, and less likely after a win. One such coalgebra might explicitly be defined as

$$c_{\text{slot}}(p) = \begin{cases} (1, \frac{p}{2}) \text{ with probability } p \\ (0, \frac{p+1}{2}) \text{ with probability } 1-p. \end{cases} \tag{5.1}$$

Iterating again returns the following distribution:

$$c^{(2)}_{\text{slot}}(p) = \begin{cases} (1,1,\frac{p}{4}) \text{ with probability } \frac{p^2}{2} \\ (1,0,\frac{p+2}{4}) \text{ with probability } \frac{p(2-p)}{4} \\ (0,1,\frac{p+1}{4}) \text{ with probability } \frac{1-p^2}{2} \\ (0,0,\frac{p+3}{4}) \text{ with probability } \frac{(1-p)^2}{2}. \end{cases} \tag{5.2}$$

Alternatively, a less attentive slot machine might just flip the same coin over and over, enacting a Bernoulli trial each time it is called and leaving the state unchanged:

$$c_{\text{bern}}(p) = \begin{cases} (1,p) \text{ with probability } p \\ (0,p) \text{ with probability } 1-p. \end{cases} \tag{5.3}$$

$c^{(n)}_{\text{bern}}(p)$ returns the tuple $(m_1, \ldots, m_n, p) \in \mathbf{2}^n \times [0,1]$ with probability $\binom{n}{k} p^k (1-p)^{n-k}$, where $k = \sum m_i$ is the number of 1s in the tuple.

Given a coalgebra $c\colon Y \rightsquigarrow X \times Y$ in $\mathcal{K}l(\mathcal{R})$, projecting the maps $c^{(n)}$ onto the X^n components gives Kleisli morphisms $k_n\colon Y \rightsquigarrow X^n$, a parametrised sequence of measures. It is not hard to show that such a sequence satisfies the conditions of the categorical Kolmogorov extension theorem, and so there is a unique morphism $k_{\mathbb{N}}\colon Y \rightsquigarrow X^{\mathbb{N}}$ making the appropriate diagrams commute. In the context of coalgebras, this realises $X^{\mathbb{N}}$ as the carrier of the structure of the final coalgebra of the functor $X \times -$.

It is meaningful to ask if we can understand exchangeability in this context too, and whether the classical categorical de Finetti theorem of chapter 4 admits a coalgebraic statement. With a well-chosen coalgebraic definition of exchangeability, the parametrised measures $k_n \colon Y \rightsquigarrow X^n$ made by projecting the iterants $c^{(n)}$ of an exchangeable coalgebras c are symmetric.

We can see already that c_{slot} above is not exchangeable, since projecting onto $\mathbf{2}^2$ in eq. (5.2) gives different probabilities of $(1,0)$ and $(0,1)$ for $p \neq \sqrt{3}-1$. c_{bern}, on the other hand, *is* exchangeable, as we would hope since it is in fact generating an i.i.d. sequence for Bernoulli trials.

If a coalgebra is exchangeable., there is a unique morphism of $\mu_k \colon Y \rightsquigarrow \mathcal{R}(X)$ arising from the categorical de Finetti theorem in $\mathcal{K}l(\mathcal{R})$ (in the case of c_{bern}, noting that $[0,1] \cong \mathcal{R}(\mathbf{2})$, it can be shown that this is the identity). Then we may ask if $\mathcal{R}(X)$ is the carrier of a coalgebra structure which makes it the final exchangeable coalgebra of the functor $X \times -$.

It is. In section 5.4 of this chapter, a de Finetti theorem of this form is proved: that there exists a coalgebra $\mathcal{R}(X) \rightsquigarrow X \times \mathcal{R}(X)$, called *the sample coalgebra*, that is the final exchangeable coalgebra of the functor $X \times - \colon \mathcal{K}l(\mathcal{R}) \to \mathcal{K}l(\mathcal{R})$.

5.2 Exchangeable Coalgebras and Affine, Commutative Monads

5.2.1 The General Setting

This section steps away from the settings of C*-algebras and the Radon monad, and is situated in the much more general setting of an affine, commutative monad $\mathcal{T}$ on a general Cartesian category $\mathbf{C}$. Recall definitions 2.12 to 2.14 of Cartesian categories, and affine commutative monads.

As discussed in section 2.5.1, affine commutative monads are very well suited for the study of probability: commutativity produces many of the behaviours expected for monads of distributions [108], whilst affineness acts like normalisation bringing us in to the realm of probability.

This final statement deserves clarification: diag. (2.9) states the definition of a monad being affine as the identity $\mathrm{id} = (\mathcal{T}(\pi_A), \mathcal{T}(\pi_B)) \circ \nabla_{A,B}$ as morphisms on $\mathcal{T}(A) \times \mathcal{T}(B)$. In the context of monads of measures, $\mathcal{T}(\pi_A) \colon \mathcal{T}(A \times B) \to \mathcal{T}(A)$ acts as $\mathcal{T}(\pi_A)(\mu_{A\times B}(U_A)) = \mu_{A\times B}(U_A \times B)$. The affine identity then asks that $\mu_A(U_A) = \mu_A \times \mu_B(U_A \times B) = \mu_A(U_A)\mu_B(B)$, requiring that $\mu_B(B) = 1$, and the same for $\mu_A(A)$. Thus, if a monad of measures is affine, it must only allow probability measures.

As such, many probability monads are examples of affine, commutative monads on Cartesian categories: the results of this section can be applied directly to the distribution monad $\mathcal{D}$ on **Set**; the Giry monad $\mathcal{G}$ on **Meas**; the Probabilistic powerdomain monad on directed-complete partial orders; the probability monad on quasi-Borel spaces; the Kantorovich monad on the category of complete metric spaces and short maps; and to our Radon monad $\mathcal{R}$ on **CH**, which will be used in section 5.3 [74, 85, 86].

Let $(\mathbf{C}, \times, \mathbf{1})$ be a Cartesian category. Let $(\mathcal{T}, \mu, \eta)$ be an affine, commutative monad on $\mathbf{C}$. We write the mediator map of $\mathcal{T}$ as $\nabla_{A,B} \colon \mathcal{T}(A) \times \mathcal{T}(B) \to \mathcal{T}(A \times B)$ and the strengths as $l_{A,B} \colon A \times \mathcal{T}(B) \to \mathcal{T}(A \times B)$ and $r_{A,B} \colon \mathcal{T}(A) \times B \to \mathcal{T}(A \times B)$.

We will continue with the practice of using zigzagged arrows to notate the morphism of $\mathcal{K}l(\mathcal{T})$, whilst using straight arrows to denote the morphisms of $\mathbf{C}$, though they might nonetheless be considered in $\mathcal{K}l(\mathcal{T})$. Specifically, if $f \colon A \to B$ is a morphism in $\mathbf{C}$, the corresponding morphism in $\mathcal{K}l(\mathcal{T})$ is $\eta_B \circ f \colon A \rightsquigarrow B$.

In the case that $\mathcal{T}$ is a probability monad, these are what we called the deterministic morphisms in $\mathcal{K}l(\mathcal{T})$. When $\mathcal{T}$ is a monad representing effects in computation (so the morphism of $\mathcal{K}l(\mathcal{T})$ model effectful computation) these lifted morphisms from $\mathbf{C}$ are the computations without effects.

The following result appears to have been somewhat folklore (as discussed by Hedges in [69]), but is very relevant for the building of Markov categories from monads, and is discussed and proved by Fritz [47, Prop. 3.1, 54, Prop 3.1].

Proposition 5.1. *There is a canonical monoidal structure on $\mathcal{K}l(\mathcal{T})$. In $\mathcal{K}l(\mathcal{T})$, the monoidal unit is also the terminal object.*

The monoidal structure is the same on objects as in $\mathbf{C}$. The tensor product of morphisms $f\colon A \to \mathcal{T}(B)$ and $g\colon A' \to \mathcal{T}(B')$ is given by the composition

$$A \times A' \xrightarrow{f\times g} \mathcal{T}(B) \times \mathcal{T}(B') \xrightarrow{\nabla_{B,B'}} \mathcal{T}(B \times B'). \tag{5.4}$$

We will notate this map by $f \boxtimes g\colon A \times A' \rightsquigarrow B \times B'$.

A digression on Markov categories A monoidal category $\mathbf{C}$ with the property that the monoidal unit is also the terminal object is called a *semicartesian monoidal category.* This property allows the notion of *discarding*, since each object A omits a unique map $!_A\colon A \rightsquigarrow \mathbf{1}$ and so there are maps $A\otimes !_A\colon A \otimes B \to A \otimes \mathbf{I} \cong A$ even when the monoidal product isn't a categorical product. If each object $A \in \mathbf{C}$ has in addition morphisms $\mathrm{copy}_A\colon A \to A \otimes A$, called the *copy maps*, such that $(A, \mathrm{copy}, !_A)$ forms a commutative comonoid, then the resulting category with structure is called a *Markov category* (Recall a commutative comonoid is just the categorical opposite to a commutative monoid. On a given A, the copy map is called the *comultiplication*, the discarding map the *counit.*)

It is in fact true that the Kleisli category of affine commutative monads form Markov categories, since with the monoidal structure in $\mathcal{K}l(\mathcal{T})$ about, we can equip copy maps by lifting the maps $(\mathrm{id}, \mathrm{id}) : A \to A \times A$ in the Cartesian category $\mathbf{C}$ [47]. Thus, the results proved in this chapter all take place in Markov categories.

Some study of de Finetti theorems have been done in Markov categories [48], with exchangeability in Markov categories also arising in the coalgebra-adjacent work of Virgo on unifilar machines in Markov categories [157], and quantum de Finetti theorems have been studied in the setting of involutive Markov categories [48, 50]. In the latter case, the so-called *de Finetti objects* which are studied are defined as limits of morphisms on the *countable Kolmogorov products* which take the place of our explicit $\mathcal{A}^{\otimes\mathbb{N}}$ and $X^{\mathbb{N}}$.

Two interesting angles of future exploration arise from this. First is whether the constructions of this chapter are exactly those of Kolmogorov products and de Finetti objects; in which case these coalgebraic methods would translate to any

Markov category with such objects. A second approach would be to ask whether the results of this chapter generalise to a wider range of Markov categories. This will be discussed in a forthcoming paper treating these results in full generality.

Returning to our setting above, with the monoidal structure defined in proposition 5.1 an object $A \in \mathbf{C}$ gives rise to an endofunctor $A \boxtimes - \colon \mathcal{K}l(\mathcal{T}) \to \mathcal{K}l(\mathcal{T})$. It takes an object B to $A \times B$, and a morphism $f \colon B \rightsquigarrow B'$ to $A \boxtimes f := \eta_A \boxtimes f \colon A \times B \rightsquigarrow A \times B'$, noting, of course, that η_A is the identity on A in $\mathcal{K}l(\mathcal{T})$.

More generally, for any $B \in \mathbf{C}$ and f a morphism in $\mathcal{K}l(\mathcal{T})$ we will write $B \boxtimes f := \eta_B \boxtimes f$ and $f \boxtimes B := f \boxtimes \eta_B$.

Definition 5.2. *A* coalgebra of the endofunctor $A \boxtimes -$ on $\mathcal{K}l(\mathcal{T})$ *is a morphism in* $\mathcal{K}l(\mathcal{T})$ *of the form* $c \colon B \rightsquigarrow A \times B$, *so a morphism* $c \colon B \to \mathcal{T}(A \times B)$ *in* $\mathbf{C}$, *for some* $B \in \mathbf{C}$. B *is called the* carrier of c.

The iterants of a coalgebra $c \colon B \rightsquigarrow A \times B$ *are the morphisms* $c^{(n)} \colon B \rightsquigarrow A^n \times B$ *for* $n \in \mathbb{N}_0$ *given iteratively by* $c^{(0)} = \eta_B \colon B \rightsquigarrow B \cong \mathbf{1} \times B$ *and* $c^{(n+1)} = (A^n \boxtimes c) \circ c^{(n)} \colon B \rightsquigarrow A^{n+1} \times B$. *In particular,* $c^{(1)} = c$.

$$\begin{tikzcd}
B \arrow[r, squiggly, "c^{(n)}"] \arrow[dr, squiggly, "c^{(n+1)}"'] & A^n \times B \arrow[d, squiggly, "A^n \boxtimes c"] \\
& A^{n+1} \times B
\end{tikzcd} \tag{5.5}$$

The projections of a coalgebra $c \colon B \rightsquigarrow A \times B$ *are the morphisms* $k_n := \mathcal{T}(\pi_{A^n}) \circ c^{(n)} \colon B \rightsquigarrow A^n$.

$$\begin{tikzcd}
B \arrow[r, squiggly, "c^{(n)}"] \arrow[dr, squiggly, "k_n"'] & A^n \times B \arrow[d, "\pi_{A^n}"] \\
& A^n.
\end{tikzcd} \tag{5.6}$$

Proposition 5.3. *Let* $c \colon B \rightsquigarrow A \times B$ *be a coalgebra. Then, for any* $n \in \mathbb{N}_0$ *and* $0 \leq k \leq n$, $c^{(n)} = \left(A^k \boxtimes c^{(n-k)}\right) \circ c^{(k)}$.

$$\begin{tikzcd}
B \arrow[r, squiggly, "c^{(k)}"] \arrow[dr, squiggly, "c^{(n)}"'] & A^k \times B \arrow[d, squiggly, "A^k \boxtimes c^{(n-k)}"] \\
& A^n \times B
\end{tikzcd} \tag{5.7}$$

Proof. We induct on n. The base case is immediate.

Now suppose eq. (5.7) holds for all $m \leq n$. If $k = n+1$ then $c^{(n+1)} = (A^{n+1} \boxtimes \eta_B) \circ c^{(n+1)} = \left(A^k \boxtimes c^{(n+1-k)}\right) \circ c^{(k)}$. Now suppose $0 \leq k \leq n$. Then

$$\begin{aligned} c^{(n+1)} &= (A^n \boxtimes c) \circ c^{(n)} \\ &= (A^n \boxtimes c) \circ \left(A^k \boxtimes c^{(n-k)}\right) \circ c^{(k)} \\ &= A^k \boxtimes \left(\left(A^{n-k} \boxtimes c\right) \circ c^{(n-k)}\right) \circ c^{(k)} \\ &= \left(A^k \boxtimes c^{(n+1-k)}\right) \circ c^{(k)} \end{aligned} \tag{5.8}$$

as desired. □

This is a technically helpful result. It is also intuitive: the definition of $c^{(n+1)}$ expressed that once we knew how to generate n results with c, we may use the updated value in B to generate the next one. This proposition says that we may split that sequence up in any other way, generating the first k results, and then the following $n-k$.

Definition 5.4 (Morphism of Coalgebras)**.** *Given two coalgebras $c_i \colon B_i \rightsquigarrow A \times B_i$, $(i = 1, 2)$, in $\mathcal{K}l(\mathcal{T})$,* a morphism of coalgebras $c_1 \to c_2$ *is a $\mathcal{K}l(\mathcal{T})$ map $\phi \colon B_1 \rightsquigarrow B_2$ such that $c_2 \circ \phi = (A \times \phi) \circ c_1$.*

$$\begin{tikzcd} B_1 \arrow[r, rightsquigarrow, "\phi"] \arrow[d, rightsquigarrow, "c_1"'] & B_2 \arrow[d, rightsquigarrow, "c_2"] \\ A \times B_1 \arrow[r, rightsquigarrow, "A \boxtimes \phi"'] & A \times B_2. \end{tikzcd} \tag{5.9}$$

This defines a category of coalgebras. The final coalgebra, *if it exists, is the final object of this category. In other words, it is a coalgebra $c_{\text{fin}} \colon B_{\text{fin}} \rightsquigarrow A \times B_{\text{fin}}$ such that for any $c \colon B \rightsquigarrow A \times B$ there is a unique coalgebra morphism $c \to c_{\text{fin}}$.*

5.2.2 The Stream Coalgebra

We may form the Kolmogorov extension theorem diagram in $\mathbf{C}$ and thus in $\mathcal{K}l(\mathcal{T})$, discarding the last element of each product:

$$A \longleftarrow A^2 \longleftarrow \ldots \longleftarrow A^n \longleftarrow . \tag{5.10}$$

Theorem 5.5 (Consistent Measures from Coalgebras)**.** *The projections* $k_n \colon B \rightsquigarrow A^n$ *of a coalgebra* $c \colon B \rightsquigarrow A \times B$ *form a cone of the Kolmogorov extension theorem diagram, diag.* (5.10)*, in* $\mathcal{K}l(\mathcal{T})$*.*

Proof. In $\mathbf{C}$, the projections $C \times D \to C$ are equivalently the maps $C \times !_D \colon C \times D \to C \times \mathbf{1} \cong C$. As such, $k_n = (A^n \times !_B) \circ c^{(n)} = (A^n \times !_A) \circ (A^{n+1} \times !_B) \circ (A^n \boxtimes c) \circ c^{(n)} = (A^n \times !_A) \circ k_{n+1}$, as desired, where the second equality follows from the fact that all introduced morphisms are the identity on A^n. □

In the following definition, we ask for a limit object $A^{\mathbb{N}}$ in $\mathbf{C}$ which lifts to a limit in $\mathcal{K}l(\mathcal{T})$. In the context of probability monads, the existence of such a limit in $\mathbf{C}$ is the existence of countably infinite products of a measure space with itself. To say the limit lifts is to say measures on this space are defined by deterministic projections onto their finite marginals. This is the Kolmogorov extension theorem. This definition, of an infinite product defined as a limit that is required to have deterministic projections, is exactly how *Kolmogorov products*, the infinite product spaces of Markov categories, are defined [53].

Definition 5.6 (Stream Coalgebra)**.** *Suppose, for an object* $A \in \mathbf{C}$*, diag.* (5.10) *has a limit in* $\mathbf{C}$*,* $A^{\mathbb{N}}$*, and this limit is preserved under the inclusion* $\mathbf{C} \to \mathcal{K}l(\mathcal{T})$*. Then* the stream coalgebra on $A \in \mathbf{C}$ *is the coalgebra* $c_{\text{stream}} \colon A^{\mathbb{N}} \to A \times A^{\mathbb{N}}$ *given by the isomorphism in* $\mathbf{C}$ (p_1, shift)*, for the projection* $p_1 \colon A^{\mathbb{N}} \to A$ *and* $\text{shift} \colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ *the unique limit morphism extending the morphisms* $(!_A \times A^n) \circ p_{n+1} \colon A^{\mathbb{N}} \to A^n$*.*

$$\begin{tikzcd}
 & A \times A^n \arrow[rr, "!_A \times A^n"] & & A^n \\
A^{\mathbb{N}} \arrow[ur, "p_{n+1}"] \arrow[dr, "p_{n+2}"'] \arrow[rr, dashed, "\text{shift}"] & & A^{\mathbb{N}} \arrow[ur, "p_n"] \arrow[dr, "p_{n+1}"'] & \\
 & A \times A^{n+1} \arrow[rr, "!_A \times A^{n+1}"'] & & A^{n+1} \arrow[uu, "\pi_{n+1}"']
\end{tikzcd} \tag{5.11}$$

The intuition here is that c_{stream} takes a stream $(x_i)_{i \geq 1} \in A^{\mathbb{N}}$ to $\big(x_1, (x_i)_{i \geq 2}\big) \in A \times$.

Theorem 5.7. *If it exists, the stream coalgebra c_{stream} is the final coalgebra of functor the $A \boxtimes -\colon \mathcal{K}l(\mathcal{T}) \to \mathcal{K}l(\mathcal{T})$.*

Proof. Let $c\colon B \rightsquigarrow A \times B$ be a coalgebra of $A \times -$. By theorem 5.5, the maps $k_n\colon B \rightsquigarrow A^n$ are a cone of the diagram in $\mathcal{K}l(\mathcal{T})$ for which $A^{\mathbb{N}}$ is the limit. As such, there exists a unique morphism $k_{\mathbb{N}}\colon B \rightsquigarrow A^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $k_n = p_n \circ k_{\mathbb{N}}$.

We show $k_{\mathbb{N}}$ is a coalgebra morphism $c \to c_{\text{stream}}$. Recall this means that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, rightsquigarrow, "k_{\mathbb{N}}"] \arrow[d, rightsquigarrow, "c"'] & A^{\mathbb{N}} \arrow[d, "c_{\text{stream}}"] \\
A \times B \arrow[r, rightsquigarrow, "A \boxtimes k_{\mathbb{N}}"'] & A \times A^{\mathbb{N}}.
\end{tikzcd} \tag{5.12}$$

Note that the morphisms $A \boxtimes k_n\colon A \times B \rightsquigarrow A^{n+1}$ are a cone of diag. (5.10) and as such there is a unique map $\phi\colon A \times B \rightsquigarrow A^{\mathbb{N}} \cong A \times A^{\mathbb{N}}$, such that $p_n \circ \phi = A \times k_n$. Since $A \boxtimes k_{\mathbb{N}}$ satisfies this, it is the unique such map.

Since $(A \boxtimes k_n) \circ c = k_{n+1}$, each triangle of the following diagram commutes. It witnesses that $p_n \bullet c_{\text{stream}}^{-1} \bullet (A \boxtimes k_{\mathbb{N}}) \circ c = k_n$ for all $n \in \mathbb{N}$, where $\circ$ for composition in $\mathcal{K}l(\mathcal{T})$ and $\bullet$ is used to denote that the morphism on the left is a morphism of $\mathbf{C}$ lifted to $\mathcal{K}l(\mathcal{T})$. As such $f \bullet g = \left[\eta_{\text{ran}(f)} \circ f\right]_{\mathbf{C}} \circ g$, where composition in the brackets is in $\mathbf{C}$.

$$\begin{tikzcd}
 & & & A^n \\
B \arrow[r, rightsquigarrow, "c"] \arrow[urrr, rightsquigarrow, bend left=40, "k_n"] \arrow[drrr, rightsquigarrow, bend right=40, "k_{n+1}"'] & A \times B \arrow[r, rightsquigarrow, "A \boxtimes k_{\mathbb{N}}"] \arrow[urr, rightsquigarrow, bend left=30, "A \boxtimes k_{n-1}"] \arrow[drr, rightsquigarrow, bend right=30, "A \boxtimes k_n"'] & A \times A^{\mathbb{N}} \arrow[r, "c_{\text{stream}}^{-1}", "\sim"'] \arrow[ur, "A \boxtimes p_{n-1}"] \arrow[dr, "A \boxtimes p_n"'] & A^{\mathbb{N}} \arrow[u, "p_n"'] \arrow[d, "p_{n+1}"] \\
 & & & A^{n+1} \arrow[uu, bend right=60, "\pi"']
\end{tikzcd} \tag{5.13}$$

Then, we must have $c_{\text{stream}}^{-1} \bullet (A \boxtimes k_{\mathbb{N}}) \circ c = k_{\mathbb{N}}$, as desired, since the limit $A^{\mathbb{N}}$ with its maps p_n is preserved in $\mathcal{K}l(\mathcal{T})$.

For uniqueness, let $\phi\colon c \to c_{\mathrm{stream}}$ be a coalgebra morphism. We show by induction that in $\mathbf{C}$

$$(A^n \times !_{A^{\mathbb{N}}}) \circ c^{(n)}_{\mathrm{stream}} = p_n \tag{5.14}$$

for all $n \in \mathbb{N}$. The $n = 1$ case follows from the definition $c_{\mathrm{stream}} = (p_1, \mathrm{shift})$. Supposing now eq. (5.14) is true for some n, then, in $\mathbf{C}$,

$$\begin{aligned}
\left(A^{n+1} \times !_{A^{\mathbb{N}}}\right) \circ c^{(n+1)}_{\mathrm{stream}} &= \left(A^{n+1} \times !_{A^{\mathbb{N}}}\right) \circ \left(A \times c^{(n)}_{\mathrm{stream}}\right) \circ c_{\mathrm{stream}} \\
&= \left(A \times \left((A^n \times !_{A^{\mathbb{N}}}) \circ c^{(n)}_{\mathrm{stream}}\right)\right) \circ c_{\mathrm{stream}} \\
&= (A \times p_n) \circ c_{\mathrm{stream}} \\
&= (p_1, p_n \circ \mathrm{shift}) \\
&= (p_1, (!_A \times A^n) \circ p_{n+1}) = p_{n+1}
\end{aligned} \tag{5.15}$$

where the first line follows from the decomposition formula for $c^{(n)}$, proposition 5.3, and the last follows from the universal properties of the product and the terminal object.

The following diagram commutes, since each square is the image of the coalgebra morphism condition in diag. (5.9) under the functor $A^n \boxtimes -$.

$$\begin{tikzcd}
B \arrow[r, rightsquigarrow, "c"] \arrow[dd, rightsquigarrow, "\phi"'] & A \times B \arrow[r, rightsquigarrow] \arrow[dd, rightsquigarrow, "A\boxtimes\phi"'] & \dots \arrow[r, rightsquigarrow] & A^{n-1} \times B \arrow[r, rightsquigarrow, "A^{n-1}\boxtimes c"] \arrow[dd, rightsquigarrow, "A^{n-1}\boxtimes\phi"'] & A^n \times B \arrow[dd, rightsquigarrow, "A^n\boxtimes\phi"'] \arrow[dr, "\pi"] & \\
& & & & & A^n \\
A^{\mathbb{N}} \arrow[r, "c_{\mathrm{stream}}"'] & A \times A^{\mathbb{N}} \arrow[r] & \dots \arrow[r] & A^{n-1} \times A^{\mathbb{N}} \arrow[r, "A^{n-1}\times c_{\mathrm{stream}}"'] & A^n \times A^{\mathbb{N}} \arrow[ur, "A^n \times !_{A^{\mathbb{N}}}"'] &
\end{tikzcd} \tag{5.16}$$

Following from B through to A^n and using eq. (5.14) gives

$$k_n = (A^n \times !_{A^{\mathbb{N}}}) \circ c^{(n)}_{\mathrm{stream}} \circ \phi = p_n \circ \phi. \tag{5.17}$$

Thus, $\phi = k_{\mathbb{N}}$ since $k_{\mathbb{N}}$ is the unique solution to these equations. □

Note that in the study of coalgebraic approaches to probability, Kerstan and König have written about similar Kolmogorov extension theorem-like final coalgebra results in [105]. These results are for $\mathcal{K}l(\mathcal{G})$ and a sub-probability monad (a monad of measures with total measure less than or equal to one), alongside also exhibiting final coalgebras for related functors describing what they call *probabilistic transition systems*, which may terminate.

5.2.3 Exchangeable Coalgebras and the Sample Coalgebra

We may also discuss exchangeability in the context of coalgebras. Though we may be tempted to do so using the projections k_n, the definition proposed by Jacobs and Staton instead treats exchangeability using an approach more informed by the idea of a coalgebra as a model for computation, keeping hold of the transformed state. Further, it uses the fact that we really only need symmetry of any two consecutive outputs from the coalgebra.

Definition 5.8 (Exchangeable Coalgebra)**.** *A coalgebra* $c\colon B \rightsquigarrow A \times B$ *is* exchangeable *if the following diagram commutes:*

$$\begin{tikzcd}
 & A^2 \times B \arrow[dd, "swap \times \mathrm{id}_B"] \\
B \arrow[ur, squiggly, "c^{(2)}"] \arrow[dr, squiggly, "c^{(2)}"'] & \\
 & A^2 \times B.
\end{tikzcd} \tag{5.18}$$

Recall the $\mathbf{Inj}^{\mathrm{op}}$-shaped de Finetti diagram of theorem 4.3. It takes every injection $\tau\colon \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ to the morphism in **CH**

$$X^\tau\colon\ (x_1, \dots, x_n) \mapsto \left(x_{\tau(1)}, \dots, x_{\tau(m)}\right). \tag{5.19}$$

Each of these maps is the composition of a projection $X^n \to X^m$ which drops the indices in $\{1, \dots, n\} \setminus \tau(\{1, \dots, m\})$ and then a permutation of the remaining indices using a braiding map.

As such, the same diagram can be constructed in **C** and in $\mathcal{K}\ell(\mathcal{T})$ using the discarding and braid maps.

$$\begin{tikzcd}
A \arrow[loop above] & A^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop, out=150, in=80, looseness=4] \arrow[loop, out=100, in=30, looseness=4] & A^3 \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop, out=150, in=110, looseness=6] \arrow[loop, out=120, in=90, looseness=6] \arrow[loop, out=90, in=60, looseness=6] \arrow[loop, out=70, in=30, looseness=6] & \vdots \arrow[l, shift left=3] \arrow[l, shift right=3] & \cdots & \vdots \arrow[l, shift left=3] \arrow[l, shift right=3] & A^n \arrow[l, shift left=3] \arrow[l, shift right=3] \arrow[loop, out=150, in=110, looseness=6] \arrow[loop, out=120, in=90, looseness=6] \arrow[loop, out=90, in=60, looseness=6] \arrow[loop, out=70, in=30, looseness=6] & \vdots \arrow[l, shift left=3] \arrow[l, shift right=3] & \cdots
\end{tikzcd} \tag{5.20}$$

Theorem 5.9 (Exchangeable coalgebras give exchangeable measures)**.** *If* $c\colon B \rightsquigarrow A \times B$ *is an exchangeable coalgebra, then the projections* $k_n\colon B \rightsquigarrow A^n$ *form a cone of the de Finetti diagram, diag. (5.20), in* $\mathcal{K}\ell(\mathcal{T})$.

Proof. The sequence k_n is compatible with discarding of the last element by theorem 5.5.

To show symmetry of these parametrised measures under the braiding maps A^σ for $\sigma \in \mathcal{S}_n$, we induct on n. It is sufficient to show this for $c^{(n)}$, by the commutativity of the following diagram in $\mathbf{C}$ for all $n \in \mathbb{N}$, $\sigma \in \mathcal{S}_n$:

$$\begin{tikzcd}
A^n \times B \arrow[r, "A^n \times !_B"] \arrow[d, "A^\sigma \times B"'] & A^n \arrow[d, "A^\sigma"] \\
A^n \times B \arrow[r, "A^n \times !_B"'] & A^n.
\end{tikzcd} \tag{5.21}$$

For $n = 1$, there is only the identity permutation.

Now, for $n \geq 1$, suppose $A^\sigma \circ k_n = k_n$ for all $\sigma \in \mathcal{S}_n$. The following diagram commutes since c is exchangeable:

$$\begin{tikzcd}
 & A^2 \times B \arrow[r, rightsquigarrow, "A^2 \times c^{(n-2)}"] \arrow[dd, "swap \times B"] & A^{n+1} \times B \arrow[dd, "swap \times A^{n-2} \times B"] \\
B \arrow[ur, rightsquigarrow, "c^{(2)}"] \arrow[dr, rightsquigarrow, "c^{(2)}"'] & & \\
 & A^2 \times B \arrow[r, rightsquigarrow, "A^2 \times c^{(n-2)}"'] & A^{n+1} \times B
\end{tikzcd} \tag{5.22}$$

and for any $\sigma \in \mathcal{S}_n$ this diagram commutes by the induction hypothesis:

$$\begin{tikzcd}
 & & A^{n+1} \times B \arrow[dd, "A \times A^\sigma \times B"] \\
B \arrow[r, rightsquigarrow, "c"] \arrow[urr, rightsquigarrow, bend left, "c^{(n+1)}"] \arrow[drr, rightsquigarrow, bend right, "c^{(n+1)}"'] & A \times B \arrow[ur, rightsquigarrow, "A \times c^{(n)}"] \arrow[dr, rightsquigarrow, "A \times c^{(n)}"'] & \\
 & & A^{n+1} \times B.
\end{tikzcd} \tag{5.23}$$

As such k_{n+1} is invariant under permutations generated by swapping the first two elements and arbitrary permutations of the final n. These generate $\mathcal{S}_{n+1}$. $\square$

Now we turn to the sample coalgebra. Recall eq. (5.3): the Bernoulli coalgebra $c_{\text{bern}} \colon [0,1] \rightsquigarrow \mathbf{2} \times [0,1]$ takes a bias $p \in [0,1]$ to the result of a coin flip with that bias and does not change the bias.

Noting that $[0,1] \cong \mathcal{R}(\mathbf{2})$, c_{bern} can be defined abstractly as the composition

$$\mathcal{R}(\mathbf{2}) \xrightarrow{(\text{id},\text{id})} \mathcal{R}(\mathbf{2}) \times \mathcal{R}(\mathbf{2}) \xrightarrow{l_{\mathcal{R}}} \mathcal{R}(\mathbf{2} \times \mathcal{R}(\mathbf{2})). \tag{5.24}$$

This composition makes sense not just for a general $X \in \mathbf{CH}$ in place of $\mathbf{2}$, but in fact for any object A in the Kleisli category $\mathcal{K}l(\mathcal{T})$ of our commutative monad $\mathcal{T}$ on the Cartesian category $\mathbf{C}$.

Definition 5.10 (Sample Coalgebra)**.** *Given $A \in \mathbf{C}$, the* sample coalgebra of the functor $A \boxtimes -$ in $\mathcal{K}l(\mathcal{T})$ *is the coalgebra $c_{\text{samp}} \colon \mathcal{T}(A) \rightsquigarrow A \times \mathcal{T}(A)$ given by the composition in* $\mathbf{C}$

$$\mathcal{T}(A) \xrightarrow{(\text{id},\text{id})} \mathcal{T}(A) \times \mathcal{T}(A) \xrightarrow{l_{A,\mathcal{T}(A)}} \mathcal{T}(A \times \mathcal{T}(A)). \tag{5.25}$$

When specified to the case of $\mathcal{R}$ on $\mathbf{CH}$, or more generally any probability monad, the sample coalgebra takes a measure $\mu \in \mathcal{R}(X)$ and then generates an element of X using μ, and returns μ unchanged. This coalgebra will be important later in this chapter where we will show it is the final exchangeable coalgebra of $X \times -$ in $\mathcal{K}l(\mathcal{R})$.

Coherence of commutative monads In theorem 5.11, we will show that $c^{(n)}_{\text{samp}}$ and k^{samp}_n have concise closed forms in terms of the structure transformations, specifically the mediator and the strength, of the monad $\mathcal{T}$ and the copy maps, $\text{copy}_n = (\text{id}, \ldots, \text{id}) \colon A \to A^n$, in $\mathbf{C}$. This will make it immediately evident that c_{samp} is exchangeable, and will be used to show that if $\mathcal{T}$ is a probability monad, and in particular if $\mathcal{T} = \mathcal{R}$, $k^{\text{samp}}_n = \text{iid}_n$. To do this we make use of a technical coherence theorem for monoidal monads of Došen and Petrić, [namely, 39, 40, $\mathcal{LS}$-coherence, p. 6]. Recall that a monoidal monad is equivalent to a commutative monad (definition 2.13).

This coherence theorem operates as follows: they construct a syntactical category $\mathcal{LS}$ ([See e.g. 38, Chap. 2]) for a commutative monad with objects being formal terms that one can construct in any monoidal category with such a monad. For example, each of

$$A, \quad TA, \quad A \otimes T^2B, \quad T\left((A \otimes T^nB) \otimes (TA \otimes C)\right) \tag{5.26}$$

are objects of $\mathcal{LS}$, where A, B and C are variable names for generating objects. The morphisms are those operations constructable from the monoidal and monad structures, so for example in the second term there is a morphism $A \otimes T^2B \to A \otimes TB$ corresponding to $A \otimes \mu$.

They construct a functor $G\colon \mathcal{LS} \to \mathbf{Fun}$, where the latter category is the full subcategory of **Set** given by the finite ordinals $\{1, \ldots, n\}$. This functor takes an object to the ordinal corresponding to the number of occurrences of the monad T in the object, so the objects in $\mathcal{LS}$ of the list above would have the following images under G:

$$\varnothing, \quad \{1\}, \quad \{1,2\}, \quad \{1, \ldots, n+2\}. \tag{5.27}$$

Each morphism on $\mathcal{LS}$ (i.e. each transformation of an object which would be valid in any monoidal category with a commutative monad) induces a map of these sets, containing the information about which occurrences of T are involved in the latter ones. For example, the image of the morphism $\mu_{A\otimes TB}\colon T^2(A \otimes TB) \to T(A \otimes TB)$ will be a function $\{1,2,3\} \to \{1,2\}$. Since the first two occurrences of T in the initial term (which are both in the T^2 at the front) are responsible for the first T in the later term, the function takes 1 and 2 to 1. The third T in the first term is untouched, and becomes the second in the second term, so 3 maps to 2.

This happens compositionally over all monad and monoidal operations (including $\nabla_T, \eta_T, l_T, \otimes$, and the associators etc. of the monoidal structure), and is more clearly expressed in the paper using string diagrams. The significant result is that the functor G is faithful: if two sequences of transformations from one term to another induce the same morphism $\{1, \ldots, n\} \to \{1, \ldots, m\}$, then these transformations are the same in any monoidal category with a monoidal monad.

Returning to our setting of a monoidal category $\mathbf{C}$ with a commutative monad $\mathcal{T}$, the following diagram, for example, must commute, since both legs of the diagram must represent the only map $\{1,2\} \to \{1\}$.

For objects $C, D, E \in \mathbf{C}$:

$$\begin{tikzcd}
 & \mathcal{T}(C) \times \mathcal{T}(D) \times E \arrow[ddl, "\nabla_{C,D} \times E"'] \arrow[dr, "l_{C,\mathcal{T}(D)\times E}"] & \\
 & & \mathcal{T}(C \times \mathcal{T}(D) \times E) \arrow[d, "\mathcal{T}(C \times l_{D,E})"] \\
\mathcal{T}(C \times D) \times E \arrow[ddr, "l_{C\times D,E}"'] & & \mathcal{T}(C \times \mathcal{T}(D \times E)) \arrow[d, "\mathcal{T}(r_{C,D\times E})"] \\
 & & \mathcal{T}^2(C \times D \times E) \arrow[dl, "\mu_{C\times D\times E}"] \\
 & \mathcal{T}(C \times D \times E) &
\end{tikzcd} \tag{5.28}$$

Another consequence is that we may talk of the unique extension of the mediator to n objects: $\nabla_n \colon \mathcal{T}(A_1) \times \cdots \times \mathcal{T}(A_n) \to \mathcal{T}(A_1 \times \cdots \times A_n)$.

With this, we are equipped to prove the following theorem about the closed form of $c^{(n)}_{\mathrm{samp}}$ and k^{samp}_n.

Theorem 5.11. *We have the following closed form for the iterants and projections of the sample coalgebra for all* $n \in \mathbb{N}$*:*

$$c^{(n)}_{\mathrm{samp}} = l_{A^n,\mathcal{T}(A)} \circ (\nabla_n \times \mathcal{T}(A)) \circ \mathrm{copy}_{n+1}, \tag{5.29}$$

and

$$k^{\mathrm{samp}}_n = \nabla_n \circ \mathrm{copy}_n \tag{5.30}$$

where $\mathrm{copy}_n = (\mathrm{id}, \ldots, \mathrm{id}) : A \to A^n$.

$$\begin{tikzcd}
\mathcal{T}(A) \arrow[r, "\mathrm{copy}_{n+1}"] \arrow[drr, bend right, "c^{(n)}_{\mathrm{samp}}"'] & (\mathcal{T}(A))^{n+1} \arrow[r, "\nabla_n \times \mathcal{T}(A)"] & \mathcal{T}(A^n) \times \mathcal{T}(A) \arrow[d, "l_{A^n,\mathcal{T}(A)}"] \\
 & & \mathcal{T}(A^n \times \mathcal{T}(A))
\end{tikzcd} \tag{5.31}$$

Proof. The equation for k^{samp}_n follows from that for $c^{(n)}_{\mathrm{samp}}$, since the following diagram commutes by naturality of the strength and the mediator.

$$\begin{tikzcd}
\mathcal{T}(A) \arrow[r, "\mathrm{copy}_{n+1}"] \arrow[dr, "\mathrm{copy}_n"'] & (\mathcal{T}(A))^n \times \mathcal{T}(A) \arrow[r, "\nabla_n \times A"] \arrow[d, "(\mathcal{T}(A))^n \times !_{\mathcal{T}(A)}"] & \mathcal{T}(A^n) \times \mathcal{T}(A) \arrow[r, "l_{A^n,\mathcal{T}(A)}"] \arrow[d, "\mathcal{T}(A^n) \times !_{\mathcal{T}(A)}"] & \mathcal{T}(A^n \times \mathcal{T}(A)) \arrow[d, "\mathcal{T}\left(A^n \times !_{\mathcal{T}(A)}\right)"] \\
& \mathcal{T}(A)^n \arrow[r, "\nabla_n"'] & \mathcal{T}(A^n) \arrow[r, equal, "l_{A^n,1}"'] & \mathcal{T}(A^n)
\end{tikzcd} \tag{5.32}$$

We prove the case for $c_{\mathrm{samp}}^{(n)}$ by induction. The base case of $n = 1$ is true by definition, noting that $\mathrm{copy}_2 = \mathrm{copy}$ and $\nabla_1 = \mathrm{id}$.

Now suppose it holds for n, then consider diag. (5.A). The square (1) commutes by naturality of the strength. The heptagon (2) commutes by coherence in the style of [39] as discussed above: both transformation from $\mathcal{T}(A)^{n+1}$ correspond to the map $\{1, \dots, n+1\} \to \{1, 2\}$ which takes $\{1, \dots, n\}$ to 1 and $n+1$ to 2. Following the left side of the diagram gives $c_{\mathrm{samp}}^{(n+1)}$, using the induction hypothesis for $c_{\mathrm{samp}}^{(n)}$, whilst the right side gives $l_{A^{n+1},\mathcal{T}(A)} \circ (\nabla_{n+1} \times \mathcal{T}(A)) \circ \mathrm{copy}_{n+2}$. This completes the proof. □

We have the following theorem as a result:

Theorem 5.12. *For any $A \in \mathbf{C}$, the sample coalgebra $c_{\mathrm{samp}} \colon \mathcal{T}(A) \rightsquigarrow A \times \mathcal{T}(A)$ is exchangeable.*

Proof. By theorem 5.11, $c_{\mathrm{samp}}^{(2)} = l_{A^2,A} \circ (\nabla_{A,A} \times \mathcal{T}(A)) \circ (\mathrm{id}, \mathrm{id}, \mathrm{id})$.

Then the following diagram exhibiting the exchangeability of $c^{(2)}$ commutes by the naturality of l and the fact that $\mathcal{T}\left(swap_{C,D}\right) \circ \nabla_{C,D} = \nabla_{C,D} \circ swap_{\mathcal{T}(C),\mathcal{T}(D)}$:

$$\begin{tikzcd}
\mathcal{T}(A) \arrow[r, "(\mathrm{id},\mathrm{id},\mathrm{id})"] \arrow[dr, "(\mathrm{id},\mathrm{id},\mathrm{id})"'] & \mathcal{T}(A)^3 \arrow[r, "\nabla_{A,A} \times \mathcal{T}(A)"] \arrow[d, "swap \times \mathcal{T}(A)"] & \mathcal{T}(A^2) \times \mathcal{T}(A) \arrow[r, "l_{A^2,A}"] \arrow[d, "\mathcal{T}(swap) \times \mathcal{T}(A)"] & \mathcal{T}(A^2 \times \mathcal{T}(A)) \arrow[d, "\mathcal{T}(swap \otimes \mathcal{T}(A))"] \\
& \mathcal{T}(A)^3 \arrow[r, "\nabla_{A,A} \times \mathcal{T}(A)"'] & \mathcal{T}(A^2) \times \mathcal{T}(A) \arrow[r, "l_{A^2,A}"'] & \mathcal{T}(A^2 \times \mathcal{T}(A)).
\end{tikzcd} \tag{5.33}$$

□

$$\begin{tikzcd}
 & & \mathcal{T}(A) \arrow[d, "\text{copy}"] \arrow[ddr, bend left, "\text{copy}_{n+2}"] & \\
 & & \mathcal{T}(A) \times \mathcal{T}(A) \arrow[dl, "l_{A,\mathcal{T}(A)}"'] \arrow[dr, "\mathcal{T}(A)\times\text{copy}_{n+1}"] & \\
 & \mathcal{T}(A \times \mathcal{T}(A)) \arrow[dr, "\mathcal{T}\left(A\times\text{copy}_{n+1}\right)"'] & (1) & {[\mathcal{T}(A)]}^{n+2} \arrow[dl, "l_{A,(\mathcal{T}(A))^{n+1}}"] \arrow[dd, equal, bend left] \\
 & & \mathcal{T}\Big(A \times {[\mathcal{T}(A)]}^{n+1}\Big) \arrow[dl, "\mathcal{T}(A\times\nabla_n\times\mathcal{T}(A))"'] & \\
 & \mathcal{T}(A \times \mathcal{T}(A^n) \times \mathcal{T}(A)) \arrow[d, "\mathcal{T}\left(A\times l_{A^n,\mathcal{T}(A)}\right)"'] & & {[\mathcal{T}(A)]}^{n+2} \arrow[ul, "l_{A,(\mathcal{T}(A))^{n+1}}"'] \arrow[dd, "\nabla_{n+1}\times\mathcal{T}(A)"] \\
 & \mathcal{T}(A \times \mathcal{T}(A^n \times \mathcal{T}(A))) \arrow[d, "\mathcal{T}\left(r_{A,A^n\times\mathcal{T}(A)}\right)"'] & (2) & \\
 & \mathcal{T}^2(A^{n+1} \times \mathcal{T}(A)) \arrow[dr, "\mu_{A^{n+1}\times\mathcal{T}(A)}"'] & & \mathcal{T}(A^{n+1}) \times \mathcal{T}(A) \arrow[dl, "l_{A^{n+1},\mathcal{T}(A)}"] \\
 & & \mathcal{T}(A^{n+1} \times \mathcal{T}(A)) &
\end{tikzcd} \tag{5.A}$$

5.3 Coalgebras and the Radon Monad

5.3.1 The Stream Coalgebra

Specialising the definitions of definition 5.6 to the case of **CH** and the Radon monad, we get the following definition:

Definition 5.13 (Stream Coalgebra)**.** The stream coalgebra *for a space* $X \in \mathbf{CH}$

is the deterministic homeomorphism

$$\begin{aligned} c_{\text{stream}} \colon X^{\mathbb{N}} &\to X \times X^{\mathbb{N}} \\ (x_i)_{i\in\mathbb{N}} &\mapsto \left(x_1, (x_i)_{i\geq 2}\right). \end{aligned} \tag{5.34}$$

Theorem 5.14. *The stream coalgebra c_{stream} is the final coalgebra of the functor $X \times - \colon \mathcal{K}l(\mathcal{R}) \to \mathcal{K}l(\mathcal{R})$.*

Proof. $\mathcal{R}$ is an affine, commutative monad on the Cartesian category **CH**. Further, **CH** has limits of the Kolmogorov extension theorem diagram, and it is preserved in $\mathcal{K}l(\mathcal{R})$. As such, this is just theorem 5.7 with $\mathbf{C} = \mathbf{CH}$ and $\mathcal{T} = \mathcal{R}$. □

5.3.2 The Sample Coalgebra

The following results remain true in the general context of the commutative, affine monad $\mathcal{T}$ on the Cartesian category **C** above if a de Finetti limit theorem of the form of theorem 4.3 can be shown to classify $\mathcal{T}(A)$ in $\mathcal{K}l(\mathcal{T})$.

We now consider the general sample coalgebra of definition 5.10 in the setting of $\mathcal{K}l(\mathcal{R})$.

Definition 5.15 (Sample Coalgebra in $\mathcal{K}l(\mathcal{R})$)**.** *Given $X \in$ **CH**, the* sample coalgebra *is the coalgebra $c_{\text{samp}} \colon \mathcal{R}(X) \rightsquigarrow X \times \mathcal{R}(X)$ of the functor $X \times (-)$ in $\mathcal{K}l(\mathcal{R})$ given by the composition*

$$\mathcal{R}(X) \xrightarrow{\text{id}\times\text{id}} \mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{l_{\mathcal{R}}} \mathcal{R}(X \times \mathcal{R}(X)). \tag{5.35}$$

By theorem 5.14, there exists a unique map $\mathcal{R}(X) \rightsquigarrow X^{\mathbb{N}}$ which is a morphism of coalgebras $c_{\text{samp}} \to c_{\text{stream}}$. We have seen this map before, it is the limit morphism of the maps $\text{iid}_n \colon \mathcal{R}(X) \to X^n$.

Theorem 5.16. iid: $\mathcal{R}(X) \rightsquigarrow X^{\mathbb{N}}$ *is the unique coalgebra morphism $c_{\text{samp}} \to c_{\text{stream}}$.*

Proof. Since iid is the extension of $\text{iid}_n \colon \mathcal{R}(X) \rightsquigarrow X^n$, and we wish to show that $k^{\text{samp}}_{\mathbb{N}} = \text{iid}$, it is necessary and sufficient to show that $k^{\text{samp}}_n = \text{iid}_n$.

By theorem 5.11, $k^{\text{samp}}_n = \nabla_n \circ \text{copy}_n$. In $\mathcal{K}l(\mathcal{R})$, this is exactly iid_n. □

The following theorem about iid will come in handy later in the chapter, but also has some intuitive strength.

Theorem 5.17. *For* $X \in \mathbf{CH}$, $\mathrm{iid}\colon \mathcal{R}(X) \rightsquigarrow X^{\mathbb{N}}$ *is monic in* $\mathcal{K}\ell(\mathcal{R})$*.*

Monomorphicity here is slightly more challenging than it seems. It is tempting to read this statement in terms of morphisms in **CH**: the continuous map of topological spaces $\mathrm{iid}\colon \mathcal{R}(X) \to \mathcal{R}\left(X^{\mathbb{N}}\right)$ is monic. Intuitively, this would say that if two measures produce the same distribution on streams of outcomes, they must be the same measure. If this were to be untrue, the frequentist probability theorist would have major issues, since this is saying exactly about how probability measures are really determined by the relative frequencies of the outcomes of repeat experiments under them. Indeed, the above map *is* monic in **CH**, since it has left inverse $\mathcal{R}(p_1)\colon \mathcal{R}\left(X^{\mathbb{N}}\right) \to \mathcal{R}(X)$, but this is not sufficient to prove the theorem. This is because composition in $\mathcal{K}\ell(\mathcal{R})$ will not look at the streams generated by one measure, but instead by elements of $\mathcal{R}(\mathcal{R}(X))$. To show iid is monic, we must look at commuting diagrams of the form

$$\begin{tikzcd}
 & \mathcal{R}(X) \arrow[dr, rightsquigarrow, "\mathrm{iid}"] & \\
Y \arrow[ur, rightsquigarrow, "\Phi_1"] \arrow[dr, rightsquigarrow, "\Phi_2"'] & & X^{\mathbb{N}} \\
 & \mathcal{R}(X). \arrow[ur, rightsquigarrow, "\mathrm{iid}"'] &
\end{tikzcd} \tag{5.36}$$

In these, both Φ_is have co-domain $\mathcal{R}(\mathcal{R}(X))$ and if we try the same trick with $\mathcal{R}(p_1)\colon \mathcal{R}\left(X^{\mathbb{N}}\right) \rightsquigarrow X$, we just end up with $\mathrm{m}_{\mathcal{R}}(\Phi_1) = \mathrm{m}_{\mathcal{R}}(\Phi_2)$, which definitely does not imply that $\Phi_1 = \Phi_2$. Luckily though, iid is exchangeable, and de Finetti, Hewitt, Savage and our theorem 4.3 are of great help here.

Proof of theorem 5.17. Suppose $\Phi_1, \Phi_2\colon Y \rightsquigarrow \mathcal{R}(X)$ are as in diag. (5.36), that is that $\mathrm{iid} \circ \Phi_1 = \mathrm{iid} \circ \Phi_2$. We wish to show that $\Phi_1 = \Phi_2$.

Composition with $p_n\colon X^{\mathbb{N}} \to X^n$ gives that $\mathrm{iid}_n \circ \Phi_1 = \mathrm{iid}_n \circ \Phi_2$ for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
 & \mathcal{R}(X) \arrow[dr, squiggly, "\mathrm{iid}"] \arrow[drr, squiggly, bend left, "\mathrm{iid}_n"] & & \\
Y \arrow[ur, squiggly, "\Phi_1"] \arrow[dr, squiggly, "\Phi_2"'] & & X^{\mathbb{N}} \arrow[r, squiggly, "p_n"] & X^n. \\
 & \mathcal{R}(X) \arrow[ur, squiggly, "\mathrm{iid}"'] \arrow[urr, squiggly, bend right, "\mathrm{iid}_n"'] & &
\end{tikzcd} \tag{5.37}$$

In particular, this is a cone of the diagram of theorem 4.3, the categorical de Finetti-Hewitt-Savage theorem. As such, there is a unique morphism $f\colon Y \to \mathcal{R}(X)$ such that $\mathrm{iid}_n \circ f = \mathrm{iid}_n \circ \Phi_i$ for all n. As such, $f = \Phi_1 = \Phi_2$.

$$\begin{tikzcd}
 & & X^n \arrow[loop right] \\
Y \arrow[urr, squiggly, bend left, "\mathrm{iid}_n \circ \Phi_i"] \arrow[r, squiggly, dashed, "\exists ! f"] \arrow[drr, squiggly, bend right, "\mathrm{iid}_{n+1} \circ \Phi_i"'] & \mathcal{R}(X) \arrow[ur, squiggly, "\mathrm{iid}_n"] \arrow[dr, squiggly, "\mathrm{iid}_{n+1}"'] & \cdots \\
 & & X^{n+1} \arrow[loop right] \arrow[uu, shift left] \arrow[uu, shift right]
\end{tikzcd} \tag{5.38}$$

$\square$

The above argument remains true for any affine commutative monad with the appropriate Kolmogorov Extension theorem and de Finetti theorem limits and shows that in such a context unique coalgebra morphism $c_{\text{samp}} \to c_{\text{stream}}$ as coalgebras of $A \boxtimes -$ is monic in $\mathcal{K}l(\mathcal{T})$.

5.4 The Coalgebraic de Finetti Theorem

Before we can prove the main result of this section, we need the following theorem.

Theorem 5.18. *Both the Giry and Radon monads satisfying the following condition: if $f\colon B \rightsquigarrow C$ is a monomorphism in the Kleisli category of the monad, then $A \boxtimes f\colon A \times B \rightsquigarrow A \times C$ is a monomorphism.*

The interpretation here is straightforward. Monicness of f is already a strong condition: given a space of parameters Λ and a process using $\lambda \in \Lambda$ to generate a random $b \in B$, and then transforming this b with f, then the distribution on

B should be recoverable from the distribution it gives on C. If now instead λ was used to generate a pair $(a, b) \in A \times B$, and then the second component was transformed using f, whilst the first was left alone, the distribution on B should again be recoverable, since no information is lost in the retaining of a.

Proof. Suppose we are in either $\mathcal{K}l(\mathcal{G})$ or $\mathcal{K}l(\mathcal{R})$.

Suppose $f\colon B \rightsquigarrow C$ is monic. This means that if $l_1, l_2\colon \Lambda \rightsquigarrow B$ are Kleisli morphisms and $f \circ l_1 = f \circ l_2$, then $l_1 = l_2$. This condition on l_1 and l_2 explicitly says that for any $\lambda \in \Lambda$ and a measurable set $U_C \subset C$,

$$\int_B f\,(U_c \,|\, b)\; l_1(\mathrm{d}b \,|\, \lambda) = \int_B f\,(U_c \,|\, b)\; l_2(\mathrm{d}b \,|\, \lambda). \tag{5.39}$$

Firstly we note that for either ϕ_i,

$$\begin{aligned}(A \boxtimes f) \circ \phi_i\,(U_A \times C \,|\, \lambda) &= \int_{A\times B} \delta_a(U_A) f\,(C \,|\, b)\; \phi_i\,(\mathrm{d}a\,\mathrm{d}b \,|\, \lambda)\\ &= \int_{A\times B} \delta_a(U_A)\, \phi_i\,(\mathrm{d}a\,\mathrm{d}b \,|\, \lambda)\\ &= \phi_i\,(U_A \times B \,|\, \lambda)\,,\end{aligned} \tag{5.40}$$

so $\phi_1\,(U_A \times B \,|\, \lambda) = \phi_2\,(U_A \times B \,|\, \lambda)$ for each $\lambda \in \Lambda$.

Now suppose $\phi_1\,(U_A \times B \,|\, \lambda) = \phi_2\,(U_A \times B \,|\, \lambda) = M_\lambda$. We need only consider $M_\lambda = 0$ and $M_\lambda \neq 0$. In the former case, for any $U_B \subset B$ measurable, we have $\phi_1\,(U_A \times U_B \,|\, \lambda) = 0 = \phi_2\,(U_A \times U_B \,|\, \lambda)$.

Now let $\tilde{\Lambda} := \{\lambda \,|M_\lambda \neq 0\} \subset \Lambda$ and take $\lambda \in \tilde{\Lambda}$. Then the map $\tilde{\phi}_i\colon \lambda \mapsto \frac{1}{M_\lambda}\phi_i\,(U_A \times - \,|\, \lambda)$ is a morphism $\tilde{\phi}_i\colon \{\lambda\} \rightsquigarrow B$ in $\mathcal{K}l(\mathcal{G})$ or $\mathcal{K}l(\mathcal{R})$.

Then, for a measurable $U_C \subset C$,

$$\begin{aligned}(A \boxtimes f) \circ \phi_i\,(U_A \times U_C \,|\, \lambda) &= \int_{A\times B} \delta_a(U_A) f\,(U_C \,|\, b)\; \phi_i\,(\mathrm{d}a\,\mathrm{d}b \,|\, \lambda)\\ &= \int_B f\,(U_C \,|\, b)\; \phi_i\,(U_A \times \mathrm{d}b \,|\, \lambda)\\ &= M_\lambda \int_B f\,(U_C \,|\, b)\; \tilde{\phi}_i\,(U_A \times \mathrm{d}b \,|\, \lambda)\,.\end{aligned} \tag{5.41}$$

Since the first term is equal for either value of i, we find that for all $\lambda \in \tilde{\Lambda}$, $\int_B f\,(U_C \,|\, b)\; \tilde{\phi}_1\,(U_A \times \mathrm{d}b \,|\, \lambda) = \int_B f\,(U_C \,|\, b)\; \tilde{\phi}_2\,(U_A \times \mathrm{d}b \,|\, \lambda)$. Thus, by monicness of f, $\tilde{\phi}_1 = \tilde{\phi}_2$, which is to say for any $U_A \subset A$ and $U_B \subset B$ measurable, $\phi_1\,(U_A \times U_B \,|\, \lambda) = \phi_2\,(U_A \times U_B \,|\, \lambda)$.

Combining the two cases for M_λ, $\phi_1(- \mid \lambda) = \phi_2(- \mid \lambda)$ for all $\lambda \in \Lambda$ and thus $\phi_1 = \phi_2$. □

With this, we are ready to prove the coalgebraic form of the Hewitt-Savage-de Finetti theorem.

Theorem 5.19 (Coalgebraic Hewitt-Savage-de Finetti Theorem)**.** *Let X be a compact Hausdorff space. The sample coalgebra $c_{\mathrm{samp}} \colon \mathcal{R}(X) \rightsquigarrow X \times \mathcal{R}(X)$ is the final exchangeable coalgebra of the endofunctor $X \times -$ on $\mathcal{K}l(\mathcal{R})$.*

Explicitly, if $c \colon Y \rightsquigarrow X \times Y$ is an exchangeable coalgebra, there exists a unique coalgebra morphism $c \to c_{\mathrm{samp}}$.

Proof. Suppose $c \colon Y \rightsquigarrow X \times Y$ is an exchangeable coalgebra. By theorem 5.14, there is a unique coalgebra morphism $k_{\mathbb{N}} \colon c \to c_{\mathrm{stream}}$.

By theorem 5.9, the maps $k_n \colon Y \rightsquigarrow X^n$ form a cone over the de Finetti limit diagram of theorem 4.3. As such, there exists a unique map $\mu_k \colon Y \rightsquigarrow \mathcal{R}(X)$ such that for all $n \in \mathbb{N}$, $k_n = \mathrm{iid}_n \circ \mu_k$. As such, by uniqueness of the limiting map to $X^{\mathbb{N}}$, we have $k_{\mathbb{N}} = \mathrm{iid} \circ \mu_k$.

Consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, squiggly, "\mu_k"] \arrow[d, squiggly, "c"'] & \mathcal{R}(X) \arrow[r, squiggly, "\mathrm{iid}"] \arrow[d, squiggly, "c_{\mathrm{samp}}"] & X^{\mathbb{N}} \arrow[d, squiggly, "c_{\mathrm{stream}}"] \\
X \times Y \arrow[r, squiggly, "X \times \mu_k"'] & X \times \mathcal{R}(X) \arrow[r, squiggly, "X \times \mathrm{iid}"'] & X \times X^{\mathbb{N}}.
\end{tikzcd} \tag{5.42}$$

The outer rectangle is diag. (5.12), the diagram describing $k_{\mathbb{N}}$ as a coalgebra morphism, and thus commutes. The right-hand square commutes by theorem 5.16.

Since iid is monic by theorem 5.17, so is $X \times \mathrm{iid}$ by theorem 5.18. Thus, the left-hand square commutes, and μ_k is a coalgebra morphism, as desired.

Now suppose a morphism $\phi \colon Y \rightsquigarrow \mathcal{R}(X)$ satisfies diag. (5.42). Then the following diagram commutes, since each square is diag. (5.42) with some number of Xs appended to the front.

$$\begin{tikzcd}
Y \arrow[r, rightsquigarrow, "c"] \arrow[dd, rightsquigarrow, "\phi"'] & X \times Y \arrow[r, rightsquigarrow] \arrow[dd, rightsquigarrow, "X \times \phi"'] & \dots \arrow[r, rightsquigarrow] & X^{n-1} \times Y \arrow[r, rightsquigarrow, "X^{n-1} \times c"] \arrow[dd, rightsquigarrow, "X^{n-1} \times \phi"'] & X^{n} \times Y \arrow[dd, rightsquigarrow, "X^{n} \times \phi"'] \arrow[dr, rightsquigarrow, "\pi"] & \\
 & & & & & X^{n} \\
\mathcal{R}(X) \arrow[r, rightsquigarrow, "c_{\mathrm{samp}}"'] & X \times \mathcal{R}(X) \arrow[r, rightsquigarrow] & \dots \arrow[r, rightsquigarrow] & X^{n-1} \times \mathcal{R}(X) \arrow[r, rightsquigarrow, "X^{n-1} \times c_{\mathrm{samp}}"'] & X^{n} \times \mathcal{R}(X) \arrow[ur, rightsquigarrow, "\pi"'] &
\end{tikzcd} \tag{5.43}$$

Following from the top left Y through to X^n gives

$$k_n = \pi \circ c^{(n)} = \pi \circ c^{(n)}_{\mathrm{samp}} \circ \phi = \mathrm{iid}_n \circ \phi. \tag{5.44}$$

By the limiting property of μ_k, we must have that $\phi = \mu_k$. □

5.4.1 Relationship to Jacobs and Staton's Coalgebraic de Finetti Theorem

As opposed to in chapter 4, the work of this chapter represents a straight generalisation of the work of Jacobs and Staton in [91, Sec. 7]. Indeed, here we have alternative proof for the main result of that section.

Theorem 5.20 (Theorem 12 of *De Finetti's Construction as a Categorical Limit*)**.** *The Bernoulli coalgebra* $c_{\mathrm{bern}} \colon [0,1] \rightsquigarrow \mathbf{2} \times [0,1]$ *is the final exchangeable coalgebra of the functor* $\mathbf{2} \times -$*, in* $\mathcal{K}l(\mathcal{G})$*.*

Note the relocation to the Kleisli category of the Giry monad. This is of no concern to us, given the generality of section 5.2.

Proof. The main result of *De Finetti's Construction* is that $\mathcal{G}(\mathbf{2}) \cong [0,1]$ is the limit of the multiset de Finetti diagram, which in this work is the diagram of theorem 4.16. They also state a Kolmogorov extension theorem limit, $\mathbf{2}^{\mathbb{N}}$, for $\mathbf{2}$ in **Meas** and $\mathcal{K}l(\mathcal{G})$.

Meas is Cartesian, and the Giry monad $\mathcal{G} \colon \mathbf{Meas} \to \mathbf{Meas}$ is affine and commutative. Using the Kolmogorov extension theorem limit above, this gives the final coalgebra as $c_{\mathrm{stream}} \colon \mathbf{2}^{\mathbb{N}} \xrightarrow{\sim} \mathbf{2} \times \mathbf{2}^{\mathbb{N}}$. Using the commutativity of $\mathcal{G}$, we construct the sample coalgebra $c_{\mathrm{samp}} \colon \mathcal{G}(\mathbf{2}) \rightsquigarrow \mathbf{2} \times \mathcal{G}(\mathbf{2})$.

The isomorphism $\mathcal{G}(\mathbf{2}) \cong [0,1]$, which takes a measure μ to $\mu(1)$ is also a morphism of coalgebras and thus gives an isomorphism $c_{\mathrm{bern}} \cong c_{\mathrm{samp}}$.

The only aspect of the proof of theorem 5.19 that doesn't immediately apply to $\mathcal{G}$ is theorem 5.17, that the morphism $\mathrm{iid}\colon \mathcal{G}(\mathbf{2}) \rightsquigarrow \mathbf{2}^{\mathbb{N}}$, which is also the unique coalgebra morphism $c_{\mathrm{samp}} \to c_{\mathrm{stream}}$, is monic. However, the proof of that theorem uses only the fact that $\mathcal{R}(X)$ and $X^{\mathbb{N}}$ are limits of the de Finetti and Kolmogorov extension theorem diagrams respectively. By the work of Jacobs and Staton, this is true also of $\mathcal{G}(\mathbf{2})$ and $\mathbf{2}^{\mathbb{N}}$, so we have that iid is monic.

Given all of this, the proof of theorem 5.32 follows through identically in this case.

Thus, $c_{\mathrm{bern}}\colon [0,1] \rightsquigarrow \mathbf{2} \times [0,1]$ is the final exchangeable coalgebra of the functor $\mathbf{2} \times -$, in $\mathcal{K}l(\mathcal{G})$. ☐

We may summarise all this as follows:

Theorem 5.21 (General Coalgebraic de Finetti Theorem)**.** *Let $\mathcal{T}$ be an affine commutative monad on a Cartesian category $\mathbf{C}$. Let A be an object of $\mathbf{C}$.*

Suppose:

1. *the limit, $A^{\mathbb{N}}$, of the Kolmogorov extension theorem diagram exists in $\mathbf{C}$*

$$\begin{tikzcd} A & A^2 \arrow[l] & \ldots \arrow[l] & A^n \arrow[l] & {} \arrow[l] \end{tikzcd} \tag{5.45}$$

   and is preserved by the inclusion $\mathbf{C} \to \mathcal{K}l(\mathcal{T})$, and

2. *the limit of the de Finetti limit diagram in $\mathcal{K}l(\mathcal{T})$*

$$\begin{tikzcd} A \arrow[loop above] & A^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop, out=150, in=100] \arrow[loop, out=80, in=30] & A^3 \arrow[l, shift left=5] \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[loop, out=160, in=120] \arrow[loop, out=115, in=95] \arrow[loop, out=85, in=65] \arrow[loop, out=60, in=20] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] & A^n \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \arrow[loop, out=160, in=120] \arrow[loop, out=115, in=95] \arrow[loop, out=85, in=65] \arrow[loop, out=60, in=20] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \end{tikzcd} \tag{5.46}$$

   is $\mathcal{T}(A)$, with morphisms $\mathcal{T}(A) \rightsquigarrow A^n$ given by $\nabla_n \circ \mathrm{copy}_n = \nabla_n \circ (\mathrm{id}, \ldots, \mathrm{id})$.‘’

3. *the functor $A \boxtimes -$ preserves monomorphisms.*

Then the sample coalgebra $c_{\mathrm{samp}}\colon \mathcal{T}(A) \rightsquigarrow A \times \mathcal{T}(A)$ is the final exchangeable coalgebra of the functor $A \boxtimes -$ in $\mathcal{K}l(\mathcal{T})$.

The monomorphism-preservation condition The addition condition in the statement of Theorem 5.21 that the functor $A \boxtimes -$ on $\mathcal{K}l(\mathcal{T})$ preserves monomorphisms is worth discussing. It is not the only condition to make the theorem true. In the context of Markov categories, constructions akin to the two limit conditions theorem 5.21 exist, called Kolmogorov products and de Finetti objects, and the definitions of both include conditions that these limits are preserved under tensoring with any object of the category. This condition, if applied to the de Finetti limits of this paper, would also suffice, since $A \boxtimes -$ is the monoidal product of $\mathcal{K}l(\mathcal{T})$ and then the same argument for the monicness of iid in Theorem 5.17 would suffice to prove that $A \boxtimes \mathrm{iid}$ was also monic.

The reason, then, why we chose instead to use the preservation of monomorphisms under $A \boxtimes -$ is twofold:

1. This condition changes the emphasis from a property of the de Finetti objects themselves, to a property of the monad.
2. The condition is naturally fulfilled by probability monads, as shown in Theorem 5.18, and has an information-theoretic interpretation in this context which we believe is an important contribution to the conversation about axiomatising probability categorically.

The strength of the result An alternative angle to explore in future work is whether the opposite implication is true: if such a final exchangeable algebra exists, does it imply the existence of de Finetti limits. On the surface, this seems unlikely, as it is not immediately clear how to turn an exchangeable sequence of measures into a coalgebra creating those measures (one candidate coalgebra for a sequence $\mu_n \colon B \rightsquigarrow A$ would be $(\mu_1, \mathrm{id}_B) \colon B \rightsquigarrow A \times B$, but the second projection would just return $\mu_1 \times \mu_1$).

Where Jacobs and Staton used direct calculation and algebra to prove their Theorem 12, here we have shown a purely categorical proof and thus enlarged the spaces to which it is applicable. If de Finetti theorem limits are shown in the

Kleisli categories of other probability monads, theorem 5.21 allows a translation of these into a result about their exchangeable coalgebras.

5.5 The Transition Algebra Quantum de Finetti Theorem

In a similar way to the discussion of quantum multisets in section 4.4, we may consider theorem 5.19 in the non-commutative setting, by defining the initial exchangeable transition algebra of the functor $\mathcal{A} \mathbin{\hat{\otimes}} -$ on $\mathbf{CSt}_{\mathrm{CPU}}$. In fact, the algebraic definitions here are often simpler than their coalgebraic counterparts earlier in chapter 5. As in section 4.4, the dual statements from that section are labelled.

Transition algebra is alternative but standard terminology for an algebra $FA \to A$ of a functor, and is used here to distinguish these from C*-algebras and Eilenberg-Moore algebras

Take your choice of either the minimal or maximal C*-tensor product, and work for the remainder of the chapter in the monoidal category $\left(\mathbf{CSt}_{\mathrm{CPU}}, \hat{\otimes}, \mathbb{C}\right)$.

$\mathcal{A} \mathbin{\hat{\otimes}} -$ is a functor on $\mathbf{CSt}_{\mathrm{CPU}}$ taking $\mathcal{B} \mapsto \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}$ and a completely positive map $f \colon \mathcal{B} \to \mathcal{B}'$ to $\mathrm{id}_{\mathcal{A}} \mathbin{\hat{\otimes}} f \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}'$. Positivity of f would not suffice here, as this latter morphism may not be positive.

A transition algebra of the functor $\mathcal{A} \mathbin{\hat{\otimes}} -$ on $\mathbf{CSt}_{\mathrm{CPU}}$ is a positive, unital map $\gamma \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$. It combines an element of $\mathcal{A}$ with an element of $\mathcal{B}$ and returns a new element of $\mathcal{B}$. This can be done iteratively, defining $\gamma_{(n)} \colon \mathcal{A}^{\otimes n} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ with $\gamma_{(0)} = \mathrm{id}_{\mathcal{B}}$ and

$$\gamma_{(n+1)} \left(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes b\right) = \gamma \left(a_1 \otimes \gamma_{(n)} \left(a_2 \otimes \cdots \otimes a_{n+1} \otimes b\right)\right). \tag{5.47}$$

Particularly, $\gamma_{(1)} = \gamma$ and, for example,

$$\gamma_{(3)}(a_1 \otimes a_2 \otimes a_3 \otimes b) = \gamma \left(a_1 \otimes \gamma \left(a_2 \otimes \gamma \left(a_3 \otimes b\right)\right)\right). \tag{5.48}$$

These are the dual maps of $c^{(n)} \colon Y \rightsquigarrow X^n \times Y$. The dual maps to $k_n \colon Y \rightsquigarrow X^n$ are given by initiating $\gamma_{(n)}$ with $1_{\mathcal{B}}$:

$$\mathcal{A}^{\otimes n} \xrightarrow{\mathcal{A}^{\otimes n} \hat{\otimes} 1_{\mathcal{B}}} \mathcal{A}^{\otimes n} \mathbin{\hat{\otimes}} \mathcal{B} \xrightarrow{\gamma_{(n)}} \mathcal{B}. \tag{5.49}$$

Revisiting eq. (5.48),

$$\chi^3(a_1 \otimes a_2 \otimes a_3) = \gamma_{(3)}(a_1 \otimes a_2 \otimes a_3 \otimes 1_{\mathcal{B}}) = \gamma\left(a_1 \otimes \gamma\left(a_2 \otimes \gamma\left(a_3 \otimes 1_{\mathcal{B}}\right)\right)\right). \tag{5.50}$$

A morphism of transition algebras is dual to a morphism of coalgebras:

Definition 5.22 (Morphism of Transition Algebras, dual to definition 5.4)**.** *Given two transition algebras* $\gamma_i \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}_i \to \mathcal{B}_i$*,* $i = 1, 2$*, in* $\mathbf{CSt}_{\mathrm{CPU}}$*,* a morphism of transition algebras $\gamma_1 \to \gamma_2$ *is a* $\mathbf{CSt}_{\mathrm{CPU}}$ *map* $\psi \colon \mathcal{B}_1 \to \mathcal{B}_2$ *such that* $\gamma_2 \circ \left(\mathcal{A} \mathbin{\hat{\otimes}} \psi\right) = \psi \circ \gamma_1$*.*

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}_1 \arrow[r, "\mathcal{A}\hat{\otimes}\psi"] \arrow[d, "\gamma_1"'] & \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}_2 \arrow[d, "\gamma_2"] \\
\mathcal{B}_1 \arrow[r, "\psi"'] & \mathcal{B}_2.
\end{tikzcd} \tag{5.51}$$

This defines a category of transition algebras. The initial transition algebra*, if it exists, is the initial object of this category. In other words, it is a transition algebra* $\gamma^{\mathrm{init}} \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B}_{\mathrm{init}} \to \mathcal{B}_{\mathrm{init}}$ *such that for any* $\gamma \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ *there is a unique morphism* $\gamma^{\mathrm{init}} \to \gamma$*.*

5.5.1 The Kolmogorov Transition Algebra

Definition 5.23 (Kolmogorov transition algebra, dual to definitions 5.6 and 5.13)**.** *The* Kolmogorov transition algebra *for a C*-algebra* $\mathcal{A}$ *is the *-homomorphism* $\gamma^{\mathrm{Kol}} \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes\mathbb{N}} \to \mathcal{A}^{\otimes\mathbb{N}}$ *extending the maps* $\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes n} \cong \mathcal{A}^{\otimes n+1} \xrightarrow{\iota_n} \mathcal{A}^{\otimes\mathbb{N}}$*.*

Intuitively, this is the map identifying $a_0 \otimes a \in \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes\mathbb{N}}$ and $a_0 \otimes a \in \mathcal{A}^{\otimes\mathbb{N}}$.

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes n-1} \arrow[rr, "\sim"] \arrow[dr, "\mathcal{A}\hat{\otimes}\iota_{n-1}"'] & & \mathcal{A}^{\otimes n} \arrow[dr, "\iota_n"] & \\
& \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes\mathbb{N}} \arrow[rr, dashed, "\gamma^{\mathrm{Kol}}"] & & \mathcal{A}^{\otimes\mathbb{N}} \\
\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes n} \arrow[rr, "\sim"'] \arrow[ur, "\mathcal{A}\hat{\otimes}\iota_n"] & & \mathcal{A}^{\otimes n+1} \arrow[ur, "\iota_{n+1}"'] &
\end{tikzcd} \tag{5.52}$$

Theorem 5.24 (Non-commutative dual to theorem 5.14)**.** *The Kolmogorov transition algebra γ^{Kol} is the initial transition algebra of the functor $\mathcal{A} \mathbin{\hat{\otimes}} -\colon \mathbf{CSt}_{\mathrm{CPU}} \to \mathbf{CSt}_{\mathrm{CPU}}$.*

Proof. Let $\gamma\colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ be a transition algebra of $\mathcal{A} \mathbin{\hat{\otimes}} -$.

Note that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}^{\otimes n} \arrow[r, "\mathcal{A}^{\otimes n} \hat{\otimes} 1_{\mathcal{B}}"] \arrow[d, "\mathcal{A}^{\otimes n} \hat{\otimes} 1_{\mathcal{A}}"'] & \mathcal{A}^{\otimes n} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[r, "\gamma_{(n)}"] & \mathcal{B} \\
\mathcal{A}^{\otimes (n+1)} \arrow[r, "\mathcal{A}^{\otimes (n+1)} \hat{\otimes} 1_{\mathcal{B}}"'] & \mathcal{A}^{\otimes (n+1)} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[u, "\mathcal{A}^{\otimes n} \hat{\otimes} \gamma"] \arrow[ur, "\gamma_{(n+1)}"'] &
\end{tikzcd} \tag{5.53}$$

The triangle on the right is the definition of $\gamma_{(n+1)}$, whilst the square on the left takes $a \in \mathcal{A}^{\otimes n}$ to $a \otimes \gamma(1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = a \otimes 1_{\mathcal{B}}$, since γ is unital. Thus, the maps χ^n form a cocone over the diagram of corollary 3.62, the quantum Kolmogorov extension theorem, and there exists a unique, completely positive map $\chi^{\mathbb{N}}\colon \mathcal{A}^{\otimes \mathbb{N}} \to \mathcal{B}$ with $\iota_n \circ \chi^{\mathbb{N}} = \chi^n$ for all $n \in \mathbb{N}$.

$\chi^{\mathbb{N}}$ is a transition algebra morphism $\gamma^{\mathrm{Kol}} \to \gamma$. Recall this means the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}} \arrow[r, "\mathcal{A} \hat{\otimes} \chi^{\mathbb{N}}"] \arrow[d, "\gamma^{\mathrm{Kol}}"'] & \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[d, "\gamma"] \\
\mathcal{A}^{\otimes \mathbb{N}} \arrow[r, "\chi^{\mathbb{N}}"'] & \mathcal{B}.
\end{tikzcd} \tag{5.54}$$

Since local elements of the form $\iota_n(a)$ are dense in $\mathcal{A}^{\otimes \mathbb{N}}$, elements of the form $a_0 \otimes \iota_n(a)$ are dense in $\in \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}}$.

Working with such elements, the clockwise direction is

$$a_0 \otimes \iota_n \mapsto a_0 \otimes \chi^{\mathbb{N}}\left(\iota_n(a)\right) = a_0 \otimes \chi^n\left(a\right) \mapsto \gamma\left(a_0 \otimes \chi^n\left(a\right)\right) = \chi^{n+1}\left(a_0 \otimes a\right) \tag{5.55}$$

whilst the anti-clockwise direction, noting that $\gamma^{\mathrm{Kol}}\left(a_0 \otimes \iota_n(a)\right) = \iota_{n+1}(a_0 \otimes a)$ by the definition of γ^{Kol} in diag. (5.52), takes us to $\chi^{\mathbb{N}}\left(\iota_{n+1}(a_0 \otimes a)\right) = \chi^{n+1}\left(a_0 \otimes a\right)$.

For uniqueness, in the same way as for theorem 5.14, the following diagram commutes

$$\begin{tikzcd}
\mathcal{B} & \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[l, "\gamma"'] & \dots \arrow[l] & \mathcal{A}^{\otimes(n-1)} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[l] & \mathcal{A}^{\otimes n} \mathbin{\hat{\otimes}} Y \arrow[l, "\mathcal{A}^{n-1}\hat{\otimes}\gamma"'] & \\
 & & & & & \mathcal{A}^{n} \arrow[ul, "\mathcal{A}^{\otimes n}\hat{\otimes}1_{\mathcal{B}}"'] \arrow[dl, "\mathcal{A}^{\otimes n}\otimes 1_{\mathcal{B}}"] \\
\mathcal{A}^{\otimes \mathbb{N}} \arrow[uu, "\psi"] & \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}} \arrow[l, "\gamma^{\mathrm{Kol}}"] \arrow[uu, "\mathcal{A}\hat{\otimes}\psi"] & \dots \arrow[l] & \mathcal{A}^{\otimes(n-1)} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}} \arrow[l] \arrow[uu, "\mathcal{A}^{n-1}\hat{\otimes}\psi"] & \mathcal{A}^{\otimes n} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}} \arrow[l, "\mathcal{A}^{n-1}\hat{\otimes}\gamma^{\mathrm{Kol}}"] \arrow[uu, "\mathcal{A}^{n}\hat{\otimes}\psi"] &
\end{tikzcd} \tag{5.56}$$

and thus $\psi = \chi^{\mathbb{N}}$ by uniqueness of the colimit map. □

5.5.2 The Evaluation Transition Algebra

The dual to the sample coalgebra is the evaluation transition algebra. It is not immediately clear why the transition algebra as will be defined is the correct one, but in this section it will be proved that this *is* the initial exchangeable transition algebra, and as such when all C*-algebras are made commutative, it is the dual map of c_{samp}. Interestingly, c_{samp} is defined using the strength of $\mathcal{R}$ but no such morphism is available in the C*-algebra case. Future work would be to explore whether the definition used here is a kind of costrength, though it's not clear for what comonad. One possible angle would be to use the work of Westerbaan, demonstrating that there exists a monad on $\mathbf{CSt}_{\mathrm{MIU}}^{\mathrm{op}}$ for which $\mathbf{CSt}_{\mathrm{CPU}}^{\mathrm{op}}$ is the Kleisli category [159]. This result is abstract in nature, it guarantees the existence of such a monad, but does not describe its form. The result below, if it is indeed the monad in disguise, might be helpful towards understanding it.

Definition 5.25 (Evaluation Transition Algebra, dual to definition 5.15)**.** *Given* $\mathcal{A} \in \mathbf{CSt}_{\mathrm{CPU}}$*, the* evaluation transition algebra *is the transition algebra* $\gamma^{\mathrm{ev}} \colon \mathcal{A} \mathbin{\hat{\otimes}} CS(\mathcal{A}) \to CS(\mathcal{A})$ *given by*

$$a \otimes \Phi \mapsto \mathrm{ev}_a \cdot \Phi = \lambda \rho \in S(\mathcal{A}).\rho(a)\Phi(\rho). \tag{5.57}$$

Proposition 5.26. *The evaluation transition algebra* γ^{ev} *is well-defined.*

Proof. To do this we need to show that $\mathrm{ev}_a \cdot \Phi \colon S(\mathcal{A}) \to \mathbb{C}$ is continuous, and that the map $a \otimes \Phi \mapsto \mathrm{ev}_a \cdot \Phi$ is positive and unital.

Continuity is simple as a product of continuous functionals, since the weak-* topology is exactly the coarsest topology such that the evaluation maps are continuous.

For preservation of unit, note $\mathrm{ev}_{1_{\mathcal{A}}} \cdot 1_{CS(\mathcal{A})}(\rho) = \rho(1_{\mathcal{A}}) \cdot 1_{CS(\mathcal{A})}(\rho) = 1$.

For positivity, let $b = \lim_n a_n^* a_n \otimes \Phi_n^* \Phi_n$. Then, since convergence in the weak-* topology is pointwise convergence,

$$\begin{aligned} \gamma^{\mathrm{ev}}(b)(\rho) &= \left(\lim_n \mathrm{ev}_{a_n^* a_n} \cdot \Phi_n^* \Phi_n\right)(\rho) \\ &= \lim_n \left(\mathrm{ev}_{a_n^* a_n} \cdot \Phi_n^* \Phi_n(\rho)\right) \\ &= \lim_n \left(\rho(a_n^* a_n) |\Phi_n(\rho)|^2\right) \in \mathbb{R}_{\geq 0} \end{aligned} \tag{5.58}$$

□

By theorem 5.24, there exists a unique positive, unital map $\mathcal{A}^{\otimes \mathbb{N}} \to CS(\mathcal{A})$ which is a morphism of transition algebras $\gamma^{\mathrm{Kol}} \to \gamma^{\mathrm{ev}}$.

Proposition 5.27 (Dual to theorem 5.16)**.** *The unique transition algebra morphism* $\gamma^{\mathrm{Kol}} \to \gamma^{\mathrm{ev}}$ *is the infinite power state-evaluation map given by*

$$\begin{aligned} \epsilon_{\mathbb{N}} \colon \mathcal{A}^{\otimes \mathbb{N}} &\to CS(\mathcal{A}) \\ a \mapsto \epsilon_{\mathbb{N}}(a) &= \lambda \rho \in S(\mathcal{A}).\rho^{\otimes \mathbb{N}}(a) \,. \end{aligned} \tag{5.59}$$

Recall, the definition of $\rho^{\otimes \mathbb{N}}$ from corollary 3.63 as the limit of the states $\rho^{\otimes n} \in S(\mathcal{A}^{\otimes n})$ under the unparametrised quantum Kolmogorov extension theorem and note $\epsilon_{\mathbb{N}}$ is the limit map under corollary 3.62 of the maps $\epsilon_n \colon \mathcal{A}^{\otimes n} \to CS(\mathcal{A})$.

Proof. It is sufficient to show that $\chi_{\mathrm{ev}}^n = \epsilon_n$. We show this by induction on n. It is trivially true for $n = 1$.

Then note that

$$\chi_{\mathrm{ev}}^{n+1}(a_0 \otimes a) = \gamma^{\mathrm{ev}}\left(a_0 \otimes \chi_{\mathrm{ev}}^n(a)\right) = \mathrm{ev}_{a_0} \cdot \chi_{\mathrm{ev}}^n(a). \tag{5.60}$$

In particular, if $\chi_{\mathrm{ev}}^{n} = \epsilon_n$, Then

$$\begin{aligned} \chi_{\mathrm{ev}}^{n+1}(a_0 \otimes \cdots \otimes a_{n+1}) &= \mathrm{ev}_{a_0} \cdot \chi_{\mathrm{ev}}^{n}\,(a_1 \otimes \cdots \otimes a_{n+1}) \\ &= \mathrm{ev}_{a_0} \cdot \mathrm{ev}_{a_1} \cdots \mathrm{ev}_{a_{n+1}} \\ &= \lambda\rho \in S(\mathcal{A}).\rho(a_0)\dots\rho(a_{n+1}) \\ &= \lambda\rho \in S(\mathcal{A}).\rho^{\otimes(n+1)}\,(a_1 \otimes \cdots \otimes a_{n+1}) \\ &= \epsilon_{(n+1)}(a_1 \otimes \cdots \otimes a_{n+1}) \end{aligned} \tag{5.61}$$

so $\chi_{\mathrm{ev}}^{n} = \epsilon_n$ on the dense set of spanned by such $a_0 \otimes \cdots \otimes a_{n+1}$s. We are done. □

Theorem 5.28 (Dual to theorem 5.17)**.** *For a C^*-algebra $\mathcal{A}$, $\epsilon_{\mathbb{N}} \colon \mathcal{A}^{\otimes\mathbb{N}} \to CS(\mathcal{A})$ is epic in* $\mathbf{CSt}_{\mathrm{CPU}}$*.*

The proof is exactly dual to that of theorem 5.17 and so is omitted, using the uniqueness inherited from the categorical quantum de Finetti theorem, theorem 3.73.

$$\begin{tikzcd}
 & & CS(\mathcal{A}) \arrow[dr, squiggly] & \\
\mathcal{A}^{\otimes n} \arrow[r, squiggly, "\iota_n"] \arrow[urr, squiggly, bend left, "\epsilon_n"] \arrow[drr, squiggly, bend right, "\epsilon_n"'] & \mathcal{A}^{\otimes\mathbb{N}} \arrow[ur, squiggly, "\epsilon_{\mathbb{N}}"] \arrow[dr, squiggly, "\epsilon_{\mathbb{N}}"'] & & \mathcal{B}. \\
 & & CS(\mathcal{A}) \arrow[ur, squiggly] &
\end{tikzcd} \tag{5.62}$$

5.5.3 Exchangeable Algebras of $\mathcal{A} \mathbin{\hat{\otimes}} -$

Definition 5.29 (Exchangeable Transition Algebra, dual to definition 5.8)**.** *A transition algebra $\gamma \colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ is* exchangeable *if the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}^{\otimes 2} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[dd, "swap \mathbin{\hat{\otimes}} \mathcal{B}"'] \arrow[dr, "\chi^2"] & \\
 & \mathcal{B}. \\
\mathcal{A}^{\otimes 2} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[ur, "\chi^2"'] &
\end{tikzcd} \tag{5.63}$$

Proposition 5.30 (Dual to theorem 5.12)**.** *For any $\mathcal{A} \in$* $\mathbf{CSt}_{\mathrm{CPU}}$*, the evaluation transition algebra γ^{ev} is exchangeable.*

Proof.

$$\chi^2_{\mathrm{ev}}\left(a_1 \otimes a_2 \otimes \Phi\right) = \mathrm{ev}_{a_1} \cdot \mathrm{ev}_{a_2} \cdot \Phi = \mathrm{ev}_{a_2} \cdot \mathrm{ev}_{a_1} \cdot \Phi = \chi^2_{\mathrm{ev}}\left(a_2 \otimes a_1 \otimes \Phi\right). \tag{5.64}$$

□

Lemma 5.31 (Dual to theorem 5.9)**.** *Let $\gamma\colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ be an exchangeable transition algebra. The maps $\chi^n\colon \mathcal{A}^{\otimes n} \to \mathcal{B}$ form a cocone over the exchangeable sequence diagram of theorem 3.73.*

Proof. Given the commutativity of diag. (5.53), which showed these maps form a cocone of the quantum Kolmogorov extension theorem diagram of theorem 3.71, it is sufficient to show invariance of the maps χ^n under the braiding maps. We again induct on $n \in \mathbb{N}$, with the $n = 1$ case trivial.

Now suppose $n \geq 2$ and χ^n is invariant under the braiding maps $\mathcal{A}^{\otimes\sigma}$ for all permutations $\sigma \in \mathcal{S}_n$. Then this means χ^{n+1} is invariant under all permutations of $\{2, \dots, n+1\}$.

The exchangeability condition of diag. (5.63) says that for all $a_1, a_2 \in \mathcal{A}$ and $b \in \mathcal{B}$

$$\gamma\left(a_1 \otimes \gamma\left(a_2 \otimes b\right)\right) = \gamma\left(a_2 \otimes \gamma\left(a_1 \otimes b\right)\right). \tag{5.65}$$

Given $a_1, \dots, a_{n+1} \in \mathcal{A}$, setting $b = \chi^{n-1}(a_3 \otimes \cdots \otimes a_{n+1})$ in eq. (5.65) gives

$$\chi^{n+1}(a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1}) = \chi^{n+1}(a_2 \otimes a_1 \otimes a_3 \otimes \cdots \otimes a_{n+1}). \tag{5.66}$$

Thus, χ^{n+1} is also invariant under $\mathcal{A}^{\otimes(1\,2)}$ and these permutations generate all of $\mathcal{S}_{n+1}$. □

5.5.4 The Transition Algebra Quantum de Finetti Theorem

Theorem 5.32 (Transition Algebra Quantum de Finetti Theorem, non-commutative dual to theorem 5.19)**.** *Let $\mathcal{A}$ be a C^*-algebra. The evaluation transition algebra $\gamma^{\mathrm{ev}}\colon \mathcal{A} \mathbin{\hat{\otimes}} CS(\mathcal{A}) \to CS(\mathcal{A})$ is the initial exchangeable transition algebra of the endofunctor $\mathcal{A} \mathbin{\hat{\otimes}} -$ on $\mathbf{CSt}_{\mathrm{CPU}}$.*

Explicitly, if $\gamma\colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{B}$ is an exchangeable transition algebra, there exists a unique transition algebra morphism $\gamma^{\mathrm{ev}} \to \gamma$.

Proof. The proof is then exactly dual to the proof of theorem 5.19. For an exchangeable transition algebra $\gamma\colon \mathcal{A} \hat{\otimes} \mathcal{B} \to \mathcal{B}$, the transition algebra map $\gamma^{\mathrm{Kol}} \to \gamma$ given by theorem 5.24 as $\chi^{\mathbb{N}}\colon \mathcal{A}^{\otimes \mathbb{N}} \to \mathcal{B}$ factorises uniquely through the morphism $\gamma^{\mathrm{Kol}} \to \gamma^{\mathrm{ev}}$ given by $\epsilon_{\mathbb{N}}$, using lemma 5.31.

This gives a positive, unital map $\nu_\chi\colon CS(\mathcal{A}) \to \mathcal{B}$. It is easily shown to be the unique transition algebra morphism $\gamma^{\mathrm{ev}} \to \gamma$ by the dual arguments to those described with diags. (5.42) and (5.43).

Just as how in the classical case we require that monic maps are preserved by $X \times -$, the proof here relies on the map $\mathcal{A} \mathbin{\hat{\otimes}} f\colon \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \to \mathcal{A} \mathbin{\hat{\otimes}} \mathbb{C}$ being surjective if $f\colon \mathcal{B} \to \mathcal{C}$ is. $\square$

$$\begin{tikzcd}
\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}^{\otimes \mathbb{N}} \arrow[r, "\mathcal{A}\hat{\otimes}\epsilon_{\mathbb{N}}"] \arrow[d, "\gamma^{\mathrm{Kol}}"'] \arrow[rr, bend left, "\mathcal{A}\hat{\otimes}\chi^{\mathbb{N}}"] & \mathcal{A} \mathbin{\hat{\otimes}} CS(\mathcal{A}) \arrow[r, "\mathcal{A}\hat{\otimes}\nu_\chi"] \arrow[d, "\gamma^{\mathrm{ev}}"] & \mathcal{A} \mathbin{\hat{\otimes}} \mathcal{B} \arrow[d, "\gamma"] \\
\mathcal{A}^{\otimes \mathbb{N}} \arrow[r, "\epsilon_{\mathbb{N}}"] \arrow[rr, bend right, "\chi^{\mathbb{N}}"'] & CS(\mathcal{A}) \arrow[r, "\nu_\chi"] & \mathcal{B}.
\end{tikzcd} \tag{5.67}$$

This concludes chapter 5, in which both quantum and classical de Finetti theorems have been adapted into results about universal coalgebras and transition algebras. In the classical case, this was done using general tools of commutative monads, and in the quantum case, by defining explicitly the evaluation transition algebra.

I can still reason—I studied mathematics, which is the madness of reason—but now I want the plasma—I want to eat straight from the placenta.

— Clarice Lispector

6
Conclusion

The goal of this work has been to use the setting of categorical probability theory to resituate a number of theorems about the exchangeability of random processes. This has been through the appeal to the universal properties that they exhibit, either via colimits, limits or the property of being initial or final.

We take a moment to contextualise these results, and as such their place amongst future work.

Research Context This work fits into the growing subject of categorical probability theory. Within this area, the results are relevant to the study of probability monads, and adjacent to the study of Markov categories. This background has been covered throughout, and more specifically reviewed in section 2.5.1.

It also is a contribution to the study of categorical foundations for quantum theory. This field is live, and wide, and we do not attempt a review here. Methods are various, drawing from and combining the direct study of categories of operator algebras and their relationships to categories of Hilbert spaces, to graphical approaches to quantum computation, or abstract axiomatisation [e.g. 21, 27, 28, 73, 81, 82, 129, 138, 153]. Much of this research is concerned with the interface between classical and quantum settings, of which this thesis is also an exploration, and where probabilistic mixing of processes arise from unmixed processes [27, 31, 70–72, 139]. The mixed quantum-classical categorical nature of the proofs in this thesis are particularly relevant with the current interest in non-commutative analogues of Markov categories [25, 29, 49, 50, 124, 125] and recent monadic approaches to categories of operator algebras [56, 57, 159].

Contractible Sequences A number of avenues for future work have been mentioned throughout this work. Another would be to explore how other forms of symmetry theorems might translate into this setting. For example, Kallenberg discusses, via Ryll-Nardzewskl [135], contractible sequences of e.g. random variables, which are those sequences for which every finite subsequence of the same length has the same law. In particular, as permutations are no longer needed, this is seemingly weaker than exchangeability. Nonetheless, these two conditions can be shown to be equivalent in the context of sequences of measures on a measurable space, and so the diagrams in this thesis could instead be indexed by order-preserving injections [102, Thm. 1.1].

Graphons A lot of this DPhil was spent also looking for a similar limit theorem in the study of graphons. For a comprehensive text on graphons, including the results below, see Lovász [116].

A graphon is a symmetric measurable map $W\colon [0,1]^2 \to [0,1]$, which may be considered as an uncountable weighted graph. Alternatively, they also provide a method for generating random graphs: to generate a graph on n nodes, simply sample n points $x_1, \dots, x_n$ uniformly in $[0,1]$ and draw an edge between x_i and x_j (for $i \neq j$) with probability $W(x_i, x_j)$. This method produces a measure μ_n on the sets G_n of labelled graphs on n nodes. Such a measure is symmetric under permutation of the labels of nodes. In addition, μ_n can be obtained from μ_{n+1} by pushing forward μ_{n+1} by the map $G_{n+1} \to G_n$ dropping the $n+1^{\text{th}}$ node.

There are clear parallels here with the measures of the classical de Finetti theorems. Such a sequence of measures on graphs is called an *exchangeable random graph*, and the theory of random graphs has a similar classification theorem for these objects. Exchangeable random graphs that are generated by a graphon by the process above are called *local*, and the classification theorem says that an exchangeable random graph is a mixture of local ones. The set of graphons $\mathcal{G}$ can be given the structure of a measurable space such that this statement says for each exchangeable random graph, that is a sequence μ_n of measures on the discrete spaces G_n with compatibility conditions, there exists a measure Φ on $\mathcal{G}$ such that each μ_n is equal to choosing a graphon at random with Φ and then generating a graph on n nodes from it.

All the spaces of graphs G_n are finite and discrete. The space of graphons $\mathcal{G}$ is in fact a topological space, and is compact and Hausdorff. By every measure, this looks like a limit theorem in $\mathcal{K}l(\mathcal{R})$. The result is still promising, and we would be very interested to pursue this in the future but, in contrast to the established results of this thesis, we have been unable to find any convenient categorical route, via the quantum setting or otherwise, towards a proof. The challenge is proving the continuity of the resulting mediating map $Y \rightsquigarrow \mathcal{G}$ from a compatible parametrised random exchangeable graph $Y \rightsquigarrow G_n$. It is, instead, just direct analysis. This reveals the power of the categorical picture established in this work.

Summary of the Main Results Chapter 3 introduced the setting of categories of C*-algebras and categories associated with the Radon monad. The main result of this chapter was theorem 3.72, the categorical quantum de Finetti theorem in $\mathbf{CSt}_{\mathrm{CPU}}$, stating that the space $CS(\mathcal{A})$ has the universal property of a colimit of the exchangeability diagram of the C*-powers $\mathcal{A}^{\otimes n}$ under either the minimal or maximal C*-tensor norms.

Chapter 4 studied the specialisation of this result to commutative C*-algebras, and as such classical probability, and the extension of these methods to multisets, in the classical case, and symmetric tensors in the quantum one. The main results in this chapter were three categorical de Finetti theorems, theorems 4.3, 4.16 and 4.22. The first two showed that $\mathcal{R}(X)$, the space of Radon measures on a compact Hausdorff space X, is the limit of exchangeability diagrams in $\mathcal{K}l(\mathcal{R})$, with exchangeability expressed either directly with braiding maps or via multisets. The final theorem is a multiset analogue for the quantum de Finetti theorem, again giving $CS(\mathcal{A})$ as a colimit, but this time of the subspaces of $\mathcal{A}^{\otimes n}$ of symmetric tensors, $\mathcal{Q}_n(\mathcal{A})$.

Chapter 5 adapted these universal properties to coalgebras in the classical context, and transition algebras in the quantum context. The classical setting was mostly explored for coalgebras of $A \times -$ in the Kleisli category of a general affine, commutative monad on a Cartesian category. Specialising to the Radon monad gave theorem 5.19, the coalgebraic de Finetti theorem, where the sample coalgebra $c_{\mathrm{samp}} \colon \mathcal{R}(X) \rightsquigarrow X \times \mathcal{R}(X)$ is proved to be the final exchangeable coalgebra of the functor $X \times -$ on $\mathcal{K}l(\mathcal{R})$. The full generality of the setting is laid out in theorem 5.21. In addition, the evaluation transition algebra $\gamma^{\mathrm{ev}} \colon \mathcal{A} \mathbin{\hat{\otimes}} CS(\mathcal{A}) \to CS(\mathcal{A})$ is shown to be the initial exchangeable transition algebra of the functor $\mathcal{A} \mathbin{\hat{\otimes}} -$ on $\mathbf{CSt}_{\mathrm{CPU}}$ with the monoidal structures corresponding to either the minimal or maximal tensor products in theorem 5.32.

Alongside these larger results, a number of other significant developments have been covered, including various categorical Kolmogorov extension theorems (theorems 3.71 and 4.2), the reflection and preservation of limits by the state-space functor (theorem 3.68), closed form and exchangeability for the sample coalgebra of a general affine, commutative monad (theorems 5.11 and 5.12), and the topological extension of the study of multisets (definition 4.5).

Index of Notation

Index of Definitions

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